

Abstract graph-like space and vector-valued metric graphs

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In this note we present some abstract ideas how one can construct spaces from building blocks according to a graph. The coupling is expressed via boundary pairs, and can be applied to very different spaces such as discrete graphs, quantum graphs or graph-like manifolds. We show a spectral analysis of graph-like spaces, and consider as a special case vector-valued quantum graphs. Moreover, we provide a prototype of a convergence theorem for shrinking graph-like spaces with Dirichlet boundary conditions.

Dedicated to Pavel Exner's 70th birthday.

Keywords: abstract boundary value problems, Dirichlet-to-Neumann operator, graph Laplacians, coupled spaces

Prologue. I got interested in graph-like spaces by a question of Vadim Kostrykin, asking whether a Laplacian on a family of open sets $(X_\varepsilon)_{\varepsilon>0}$, converging to a metric graph X_0 converges to some suitable Laplacian on X_0 . At that time, I was not aware of the work of Kuchment and Zeng [KuZ01] and wrote down some ideas. Somehow Pavel must have heard about this; he invited me to visit him in Řež in October 2002, just two months after the big flood, which covered even the high-lying tracks with water, resulting in a very reduced schedule. At that time one had to buy the local ticket at *Praha Masarykovo nádraží* at a counter where one was forced to pronounce the most complicated letter in Czech language, the „Ř“ in „Řež“. At least I got the ticket I wanted, and enjoyed staying in this little pension *Hudec*. Řež at night has something very special and rare nowadays in our noise-polluted world — *Silence!* Only the dogs bark and from time to time, trains pass by on the other side of the *Vltava* ... Also Řež was a good opportunity to pick up some Czech words, as people in that little village only spoke Czech (and sometimes a little bit German) „*Máte smažený sýr?*“ — „*Dobrou chuť!*“ — „*Pivo, prosím*“ ... This invitation was the start of a very fruitful collaboration with Pavel over many years, resulting in several publications [EP05, EP07, EP09, EP13]. Pavel inspired my research on graph-like spaces, resulting even in an entire book [P12]. Pavel sometimes cites it with the words “... and then we apply the heavy German machinery ...”

Dear Pavel, thank you for having supported me over all the time; I hope you will find this new piece of “heavy German machinery” useful for our future collaboration, and that we can continue working together for a long time ...

Všechno nejlepší k narozeninám, Pavle!

1 Introduction

The present little note shall serve as a unified approach how to work on spaces that can be decomposed into building blocks (the *analytic* viewpoint) or that can be built up from building blocks (the *synthetic* viewpoint) according to a graph. We will call such spaces (*abstract*) *graph-like spaces*. They can be obtained in basically two different ways, depending whether the graph-like space is decomposed into pieces indexed by *vertices* or *edges*, respectively. We call them *vertex-coupled* or *edge-coupled*, respectively. There is also a mixed case, when one has a decomposition into parts indexed by vertices and edges (like for thin ε -neighbourhoods of embedded graphs or graph-like manifolds in the spirit of [P12]). This case can be reduced to the vertex-coupled case by considering the *subdivision graph* as underlying graph (see Definition 2.1 in Section 2.1 for details).

In the edge-coupled case, one can also choose a suitable subspace at each vertex determining the vertex conditions, very much in the spirit of a quantum graph. Indeed, one can consider edge-coupled spaces as *general* or *vector-valued quantum graphs* (see [Pa06] and also [vBM13] for a another point of view). Explaining the concept of metric and quantum graphs in an article dedicated to Pavel would be (in his own words ...) *to bring owls to Athens* or *coal to Newcastle* or *firewood to the forest* ... instead we refer to the book of Berkolaiko and Kuchment [BK13] or to [P12, Sec. 2.2]). We define the coupling via the language of abstract boundary value problems. Such a theory has been developed mostly for operators, in order to describe (all) self-adjoint extensions of a given minimal operator. As we are interested only in “geometric” non-negative operators such as Laplacians we find it more suitable to start with the corresponding quadratic or energy forms. A theory of abstract boundary value problems expressed entirely in terms of quadratic forms has been developed recently under the name *boundary pairs* in [P16], and under the name *boundary maps* in [P12] (see also [P16] and references therein for related concepts, as well as [HdSS12], especially Ch. 3 by Arlinskii). In particular, one has an abstract Dirichlet and Neumann operator, a solution operator for the Dirichlet problem and a Dirichlet-to-Neumann operator, see Section 2.

The coupling of abstract boundary value problems in Section 3 is — of course — not new (see e.g. Ch. 7 in [HdSS12] and references therein). For our graph-like spaces, the new point is the interpretation of the coupled operators such as the Neumann or Dirichlet-to-Neumann operator as a discrete vector-valued graph Laplacian.

In Section 4 of this note, we explain the concept of a distance of two abstract graph-like spaces based on their building blocks (such as the vertex or edge part of a graph-like space). This concept can be used to show convergence of a family of abstract boundary value problems to a limit one. The motivation is to give a unified approach for the convergence of many types of (concrete) graph-like spaces such as thick graphs, ε -neighbourhoods of embedded graphs or graph-like manifolds, including different types of boundary conditions (Neumann, Dirichlet).

I’d like to thank the anonymous referee for very carefully reading this manuscript, valuable suggestions and pointing out quite a lot of typos. I’m afraid there are still some left ...

2 Preliminaries

In this section we fix the notation and collect briefly some facts on discrete graphs, as well as on abstract boundary value problems (boundary pairs) and convergence of operators acting in different Hilbert spaces.

2.1 Discrete Graphs

Let $G = (V, E, \partial)$ be a countable graph, i.e., V and E are disjoint and at most countable sets and $\partial: E \rightarrow V \times V$ is a map defining the incidence between edges and vertices, namely, $\partial e = (\partial_- e, \partial_+ e)$ is the pair of the *initial* resp. *terminal vertex* of a given edge $e \in E$. Let $E(V_1, V_2) := \{e \in E \mid \partial_- e \in V_1, \partial_+ e \in V_2 \text{ or } \partial_+ e \in V_1, \partial_- e \in V_2\}$ for $V_1, V_2 \subset V$. We denote by $E_v = E(\{v\}, V) \subset E$ the set of edges adjacent with the vertex $v \in V$ and call the number $\deg v := |E_v|$ the *degree* of a vertex $v \in V$. We always assume that the graph is locally finite, i.e., that $\deg v < \infty$ for all $v \in V$ (but not necessarily uniformly bounded). For ease of notation, we also assume that the graph has no loops, i.e., edges e with $\partial_- e = \partial_+ e$.

We use the convention that we have chosen already an *orientation* of each edge via $\partial e = (\partial_- e, \partial_+ e)$, i.e., for each edge e there is not automatically an edge in E with the opposite direction. In particular, we assume that

$$\sum_{v \in V} \sum_{e \in E_v} a_e(v) = \sum_{e \in E} \sum_{v = \partial_{\pm} e} a_e(v) \quad (2.1)$$

holds for any numbers $a_e(v) \in \mathbb{C}$, and this also implies that $\sum_{v \in V} \deg v = 2|E|$ by setting $a_e(v) = 1$. We make constant use of this reordering in the sequel.

Given a graph $G = (V, E, \partial)$, we construct another graph by introducing a new vertex on each edge:

2.1 Definition. Let $G = (V, E, \partial)$ a graph. The *subdivision graph* $SG = (A, B, \tilde{\partial})$ is the graph with vertex set $A = V \cup E$ (disjoint union) and edge set $B = \bigcup_{v \in V} \{v\} \times E_v$. Moreover,

$$\tilde{\partial}: B \rightarrow A \times A, \quad b = (v, e) \mapsto \begin{cases} (\tilde{\partial}_- b, \tilde{\partial}_+ b) = (v, e), & v = \partial_- e \\ (\tilde{\partial}_- b, \tilde{\partial}_+ b) = (e, v), & v = \partial_+ e. \end{cases}$$

2.2 Boundary pairs and abstract boundary value problems

Following a good tradition („*Was interessiert mich mein Geschwätz von gestern, nichts hindert mich, weiser zu werden* . . . “), we use a slightly different terminology than in [P12, P16]; basically, we collect *all* data involved in a boundary pair and put it into a quintuple:

2.2 Definition.

1. We say that the quintuple $\Pi := (\Gamma, \mathcal{G}, \mathfrak{h}, \mathcal{H}^1, \mathcal{H})$ is an *abstract boundary value problem* if
 - \mathfrak{h} is a closed, non-negative quadratic form densely defined in a Hilbert space \mathcal{H} ; such a form is also called *energy form*; we endow its domain $\text{dom } \mathfrak{h} = \mathcal{H}^1$ with norm given by $\|f\|_{\mathcal{H}^1}^2 = \mathfrak{h}(f) + \|f\|_{\mathcal{H}}^2$; we also say that the energy form is given by $(\mathfrak{h}, \mathcal{H}^1, \mathcal{H})$;
 - \mathcal{G} is another Hilbert space and $\Gamma: \mathcal{H}^1 \rightarrow \mathcal{G}$ is a bounded operator, called *boundary map*, such that $\mathcal{G}^{1/2} := \text{ran } \Gamma (= \Gamma(\mathcal{H}^1))$ is dense in \mathcal{G} .
2. If, in addition, $\mathcal{H}^{1,D} := \ker \Gamma$ is dense in \mathcal{H} , we say that the abstract boundary value problem Π has a *dense Dirichlet domain*.¹
3. We say that the abstract boundary value problem Π is *bounded* if Γ is surjective, i.e., if $\text{ran } \Gamma = \mathcal{G}$.

¹A pair (Γ, \mathcal{G}) is called *boundary pair associated with the quadratic form* \mathfrak{h} in [P16] if $\text{ran } \Gamma$ is dense in \mathcal{G} and $\ker \Gamma$ is dense in \mathcal{H} . If only $\text{ran } \Gamma$ is dense in \mathcal{G} , then (Γ, \mathcal{G}) is called a *generalised boundary pair* in [P16].

4. We say that the abstract boundary value problem Π is *trivial* if $\mathcal{G} = \mathcal{H}$ and $\Gamma = \text{id}$.

A typical situation is $\mathcal{H} = \mathbf{L}_2(X, \mu)$ and $\mathcal{G} = \mathbf{L}_2(Y, \nu)$, where (X, μ) and (Y, ν) are measured spaces such that $Y \subset X$ is measurable. The abstract boundary value problem has a dense Dirichlet domain iff $\mu(Y) = 0$. The abstract boundary value problem is trivial iff $(X, \mu) = (Y, \nu)$ and $\Gamma = \text{id}$.

Given an abstract boundary value problem, we can define the following objects (details can be found in [P16]):

- the *Neumann operator* H as the operator associated with \mathfrak{h} ;
- the *Dirichlet operator* H^D as the operator associated with the closed (!) form $\mathfrak{h}|_{\ker \Gamma}$ with domain $\mathcal{H}^{1,D} := \ker \Gamma$;
- the *space of weak solutions* $\mathcal{N}^1(z) = \{h \in \mathcal{H}^1 \mid \mathfrak{h}(h, f) = z\langle h, f \rangle \ \forall f \in \mathcal{H}^{1,D}\}$;
- for $z \notin \sigma(H^D)$, $\mathcal{H}^1 = \mathcal{H}^{1,D} \dot{+} \mathcal{N}^1(z)$ (direct sum with closed subspaces); in particular, the *Dirichlet solution operator* $S(z) = (\Gamma|_{\mathcal{N}^1(z)})^{-1} : \text{ran } \Gamma = \mathcal{G}^{1/2} \rightarrow \mathcal{N}^1(z) \subset \mathcal{H}^1$ is defined; we also set $S := S(-1)$, i.e., the default value of z is -1 .
- for $z \notin \sigma(H^D)$, the *Dirichlet-to-Neumann (sesquilinear) form* \mathfrak{l}_z is defined via $\mathfrak{l}_z(\varphi, \psi) = (\mathfrak{h} - z\mathbf{1})(S(z)\varphi, S(-1)\psi)$, $\varphi, \psi \in \mathcal{G}^{1/2}$;
- we endow \mathcal{H}^1 with its natural norm given by $\|f\|_{\mathcal{H}^1}^2 = \mathfrak{h}(f) + \|f\|_{\mathcal{H}}^2$;
- we endow $\mathcal{G}^{1/2}$ with the norm given by $\|\varphi\|_{\mathcal{G}^{1/2}}^2 = \mathfrak{l}_{-1}(\varphi) = \|S\varphi\|_{\mathcal{H}^1}^2$;
- if the abstract boundary value problem is bounded, then $\mathcal{G}^{1/2} = \mathcal{G}$, and the two norms are equivalent; moreover, \mathfrak{l}_z is a bounded sesquilinear form on $\mathcal{G} \times \mathcal{G}$.

For an abstract boundary value problem, one can always construct another boundary map $\Gamma' : \mathcal{W} \rightarrow \mathcal{G}$ which is defined on a subspace \mathcal{W} of $\mathcal{H}^1 \cap \text{dom } H^{\max}$, where $H^{\max} := (H^{\min})^*$ and $H^{\min} := H^D \cap H$ denote the maximal resp. minimal operator, and on which Γ' is bounded. Moreover, one has the following *abstract Green's (first) formula*

$$\mathfrak{h}(f, g) = \langle H^{\max} f, g \rangle_{\mathcal{H}} + \langle \Gamma' f, \Gamma g \rangle_{\mathcal{G}} \quad (2.2)$$

for all $f \in \mathcal{W}$ and $g \in \mathcal{H}^1$.

Another property is important (see [P16] for details):

2.3 Definition. We say that an abstract boundary value problem Π (or the boundary pair (Γ, \mathcal{G})) is *elliptically regular* if the associated Dirichlet solution operator $S := S(-1) : \mathcal{G}^{1/2} \rightarrow \mathcal{H}^1$ extends to a bounded operator $\overline{S} : \mathcal{G} \rightarrow \mathcal{H}$, or equivalently, if there exists a constant $c > 0$ such that $\|S\varphi\|_{\mathcal{H}} \leq c\|\varphi\|_{\mathcal{G}}$ for all $\varphi \in \mathcal{G}^{1/2}$.

All our abstract boundary value problems treated in this note will be elliptically regular. They have the important property that the Dirichlet-to-Neumann form \mathfrak{l}_z is *closed* as form in \mathcal{G} with domain $\text{dom } \mathfrak{l}_z = \mathcal{G}^{1/2} = \text{ran } \Gamma$, and hence is associated with a closed operator $\Lambda(z)$, called *Dirichlet-to-Neumann operator*; moreover, the domain $\mathcal{G}^1 := \text{dom } \Lambda(z)$ of $\Lambda(z)$ is independent of $z \in \mathbb{C} \setminus \sigma(H^D)$. Another important consequence is the following formula on the difference of resolvents: Let $z \in \mathbb{C} \setminus (\sigma(H) \cup \sigma(H^D))$, then

$$(H - z)^{-1} = (H^D - z)^{-1} + \overline{S}(z)\Lambda(z)^{-1}\overline{S}(\overline{z})^*. \quad (2.3)$$

As a consequence of (2.3), one has e.g. the spectral characterisation

$$\lambda \in \sigma(H) \iff 0 \in \sigma(\Lambda(\lambda)) \quad (2.4)$$

for all $\lambda \in \mathbb{R} \setminus \sigma(H^D)$.

2.4 Examples. Important examples of elliptically regular abstract boundary value problems are the following:

1. Let (X, g) be a Riemannian manifold with compact smooth boundary (Y, h) , then

$$\Pi = (\Gamma, L_2(Y, h), \mathfrak{h}, H^1(X, g), L_2(X, g))$$

is an elliptically regular abstract boundary value problem with dense Dirichlet domain. Here, $\Gamma f = f|_Y$ is the Sobolev trace, and the energy form is $\mathfrak{h}(f) = \int_X |df|_g^2 d\text{vol}_g$.

This example is actually the godfather of the above-mentioned names for the derived objects: e.g. the Dirichlet resp. Neumann operators are actually the Dirichlet and Neumann Laplacians, the Dirichlet solution operator is the operator solving the Dirichlet problem (also called *Poisson operator*), the abstract Green's formula (2.2) is the usual one with $\Gamma' f$ being the normal outwards derivative and $\mathscr{W} = H^2(X)$ e.g., and the Dirichlet-to-Neumann operator has its standard interpretation.

2. Bounded abstract boundary value problems (i.e., abstract boundary value problems, where $\text{ran } \Gamma = \mathscr{G}$, or equivalently, where the Dirichlet-to-Neumann operator is bounded), and in particular abstract boundary value problems with finite dimensional boundary space \mathscr{G} , are elliptically regular.
3. Let $G = (V, E, \partial)$ be a graph. For simplicity, we consider only the normalised Laplacian here. We define an energy form via

$$\mathfrak{h}(f) = \sum_{e \in E} |f(\partial_+ e) - f(\partial_- e)|^2$$

for $f \in \mathscr{H}^1 = \mathscr{H} = \ell_2(V, \text{deg})$, where $\|f\|_{\ell_2(V, \text{deg})}^2 = \sum_{v \in V} |f(v)|^2 \text{deg } v$. Using (2.1) it is not hard to see that $0 \leq \mathfrak{h}(f) \leq 2\|f\|_{\ell_2(V, \text{deg})}^2$. The *boundary* of G is just an arbitrary non-empty subset ∂V of V (in particular, the degree of a “boundary vertex” can be arbitrary). Set $\mathscr{G} = \ell_2(\partial V, \text{deg})$ and $\Gamma f = f|_{\partial V}$. Then $\Pi = (\Gamma, \ell_2(\partial V, \text{deg}), \mathfrak{h}, \ell_2(V, \text{deg}), \ell_2(V, \text{deg}))$ is an elliptically regular abstract boundary value problem without dense Dirichlet domain (see [P16, Sec. 6.7]).

The Neumann operator acts as

$$(Hf)(v) = (\Delta_G f)(v) := \frac{1}{\text{deg } v} \sum_{e \in E_v} (f(v) - f(v_e)) \quad (2.5)$$

for $v \in V$, where v_e denotes the vertex adjacent with e and opposite to v . The Dirichlet operator acts in the same way on $\ell_2(\mathring{V}, \text{deg})$ where $\mathring{V} := V \setminus \partial V$ are the *interior* vertices (note that the Dirichlet Laplacian is not the Laplacian on the subgraph $\mathring{G} := (\mathring{V}, \mathring{E}, \mathring{\partial})$ with $\mathring{E} := E(\mathring{V}, \mathring{V})$ and $\mathring{\partial} := \partial|_{\mathring{E}}$, as the degree is still calculated in the entire graph G and not in \mathring{G}).

Moreover, the decomposition $\mathscr{H} = \ell_2(V, \text{deg}) = \ell_2(\partial V, \text{deg}) \oplus \ell_2(\mathring{V}, \text{deg}) = \mathscr{G} \oplus \ker \Gamma$ yields a block structure for H , namely,

$$H = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$$

with $A: \mathscr{G} \rightarrow \mathscr{G}$, $B: \ker \Gamma \rightarrow \mathscr{G}$ and Dirichlet operator $D = H^D: \ker \Gamma \rightarrow \ker \Gamma$. The Dirichlet-to-Neumann operator is

$$\Lambda(z) = (A - z) - B(D - z)^{-1}B^*$$

provided $z \notin \sigma(H^D) = \sigma(D)$. Moreover, the second boundary map $\Gamma': \mathscr{W} = \ell_2(V, \deg) \longrightarrow \mathscr{G} = \ell_2(\partial V, \deg)$ in Green's formula (2.2) is here

$$(\Gamma' f)(v) = \frac{1}{\deg v} \sum_{e \in E_v} (f(v) - f(v_e)), \quad v \in \partial V,$$

for $f \in \mathscr{W} = \ell_2(V, \deg)$, or in block structure, $\Gamma' = (A, B)$.

Note that we have not excluded the extreme (or trivial) case $\partial V = V$ leading to a trivial abstract boundary value problem with $\Gamma = \text{id}_{\ell_2(V)}$. In this case, $\ker \Gamma = \{0\}$, hence $A = H$, $B = 0$, $H^D = D = 0$ and $\sigma(H^D) = \emptyset$. Moreover, $\Lambda(z) = H - z$.

4. Let X be a metric graph (with underlying discrete graph $G = (V, E, \partial)$ and edge length function $\ell: E \longrightarrow (0, \infty)$, $e \mapsto \ell_e$, (see e.g. [BK13] or [P12, Sec. 2.2]) such that $\ell_0 = \inf_{e \in E} \ell_e > 0$. A bounded (hence elliptically regular) abstract boundary value problem is given by $\Pi = (\Gamma, \ell_2(V, \deg), \mathfrak{h}, H^1(X), L_2(X))$, where $\Gamma f = f|_V$ is the restriction of functions on X to the set of vertices, $\mathfrak{h}(f) = \int_X |f'(x)|^2 dx = \sum_{e \in E} \int_0^{\ell_e} |f'_e(x_e)|^2 dx_e$ and $f \in H^1(X) = \bigoplus_{e \in E} H^1([0, \ell_e]) \cap C(X)$. In this case, the Neumann operator H is the Laplacian with *standard* or (*generalised*) *Neumann* or *Kirchhoff*² *vertex conditions* and the Dirichlet operator H^D is the direct sum of the Dirichlet Laplacians on the intervals $[0, \ell_e]$, hence decoupled (see [P08, P12] for details).

2.3 Convergence of abstract boundary value problems acting in different spaces

We now define a concept of a “distance” δ for objects of abstract boundary value problems Π and $\tilde{\Pi}$ acting in different spaces. One can think of $\tilde{\Pi}$ as being a perturbation of Π , and δ measures quantitatively, how far away $\tilde{\Pi}$ is from being isomorphic with Π (see Example 2.10 below for the case $\delta = 0$). The term “convergence” refers to the situation where we consider a family $(\Pi_\varepsilon)_{\varepsilon \geq 0}$ of abstract boundary value problems; one can think of $\tilde{\Pi} = \Pi_\varepsilon$ and $\Pi = \Pi_0$ with “distance” δ_ε . If $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ then we say that Π_ε *converges to* Π_0 . Details of this concept of a “distance” between operators acting in different spaces can also be found in [P12, Ch. 4].

To be more precise, let $\Pi = (\Gamma, \mathscr{G}, \mathfrak{h}, \mathscr{H}^1, \mathscr{H})$ and $\tilde{\Pi} = (\tilde{\Gamma}, \tilde{\mathscr{G}}, \tilde{\mathfrak{h}}, \tilde{\mathscr{H}}^1, \tilde{\mathscr{H}})$ be two abstract boundary value problems. Recall that \mathscr{H}^1 is the domain of a closed non-negative form \mathfrak{h} in the Hilbert space \mathscr{H} , and that $\Gamma: \mathscr{H}^1 \longrightarrow \mathscr{G}$ is bounded with dense range, and similarly for the tilded objects. We need bounded operators

$$J: \mathscr{H} \longrightarrow \tilde{\mathscr{H}}, \quad J': \tilde{\mathscr{H}} \longrightarrow \mathscr{H}, \quad I: \mathscr{G} \longrightarrow \tilde{\mathscr{G}} \quad \text{and} \quad I': \tilde{\mathscr{G}} \longrightarrow \mathscr{G}, \quad (2.6a)$$

called *identification operators* which replace unitary or isomorphic operators. The quantity $\delta > 0$ used later on measures how far these operators differ from isomorphisms. We also need *identification operators* on the level of the energy form domains, namely

$$J^1: \mathscr{H}^1 \longrightarrow \tilde{\mathscr{H}}^1 \quad \text{and} \quad J'^1: \tilde{\mathscr{H}}^1 \longrightarrow \mathscr{H}^1. \quad (2.6b)$$

²When one calls these vertex conditions “*Kirchhoff*” as a coauthor of Pavel, one always ends up with at least a footnote (as in my first collaboration with Pavel [EP05]). For Pavel, the current conservation usually associated with this name, refers to the *probability* current, which is preserved for any self-adjoint vertex condition. Many other authors think of a more naive current, defined by a derivative considered as vector field.

In contrast to [BP16] we will not assume in this note that the identification operators I and I' on the boundary spaces \mathcal{G} and $\tilde{\mathcal{G}}$ also respect the form domains $\mathcal{G}^{1/2}$ and $\tilde{\mathcal{G}}^{1/2}$ of the Dirichlet-to-Neumann operators.

We start with the energy forms and boundary maps:

2.5 Definition. Let $\delta > 0$. We say that the energy forms \mathfrak{h} and $\tilde{\mathfrak{h}}$ are δ -close if there are identification operators J^1 and J'^1 as in (2.6) such that

$$|\tilde{\mathfrak{h}}(J^1 f, u) - \mathfrak{h}(f, J'^1 u)| \leq \delta \|u\|_{\tilde{\mathcal{H}}^1} \|f\|_{\mathcal{H}^1}$$

holds for all $f \in \mathcal{H}^1$ and $u \in \tilde{\mathcal{H}}^1$.

2.6 Definition. Let $\delta > 0$. We say that the boundary maps Γ and $\tilde{\Gamma}$ are δ -close if there exist identification operators J^1 , J'^1 , I and I' as in (2.6) such that

$$\|(I\Gamma - \tilde{\Gamma}J^1)f\|_{\tilde{\mathcal{G}}} \leq \delta \|f\|_{\mathcal{H}^1} \quad \text{and} \quad \|(I'\tilde{\Gamma} - \Gamma J'^1)u\|_{\mathcal{G}} \leq \delta \|u\|_{\tilde{\mathcal{H}}^1}$$

hold for all $f \in \mathcal{H}^1$ and $u \in \tilde{\mathcal{H}}^1$.

So far, we have only dealt with forms and their domains. Let us now define the following compatibility between the identification operators on the Hilbert space and the energy form level:

2.7 Definition. We say that the identification operators J , J' , J^1 and J'^1 are δ -quasi-unitarily equivalent with respect to the energy forms \mathfrak{h} and $\tilde{\mathfrak{h}}$ if

$$\begin{aligned} |\langle Jf, u \rangle_{\tilde{\mathcal{H}}} - \langle f, J'u \rangle_{\mathcal{H}}| &\leq \delta \|f\|_{\mathcal{H}} \|u\|_{\tilde{\mathcal{H}}}, \\ \|f - J'Jf\|_{\mathcal{H}} &\leq \delta \|f\|_{\mathcal{H}^1}, \quad \|u - JJ'u\|_{\tilde{\mathcal{H}}} \leq \delta \|u\|_{\tilde{\mathcal{H}}^1}, \\ \|J^1 f - Jf\|_{\tilde{\mathcal{H}}} &\leq \delta \|f\|_{\mathcal{H}^1} \quad \text{and} \quad \|J'^1 u - J'u\|_{\mathcal{H}} \leq \delta \|u\|_{\tilde{\mathcal{H}}^1} \end{aligned}$$

hold for f and u in the respective spaces. We say that the forms \mathfrak{h} and $\tilde{\mathfrak{h}}$ are δ -quasi-unitarily equivalent, if they are δ -close with δ -quasi-unitarily equivalent identification operators.

For the boundary identification operators I and I' we define:

2.8 Definition. We say that the identification operators I and I' are δ -quasi-isomorphic with respect to the abstract boundary value problems Π and $\tilde{\Pi}$ if

$$\|\varphi - I'I\varphi\|_{\mathcal{H}} \leq \delta \|\varphi\|_{\mathcal{G}^{1/2}}, \quad \|\psi - II'\psi\|_{\tilde{\mathcal{H}}} \leq \delta \|\psi\|_{\tilde{\mathcal{G}}^{1/2}}$$

hold for $\varphi \in \mathcal{G}^{1/2}$ and $\psi \in \tilde{\mathcal{G}}^{1/2}$. We say that the boundary maps Γ and $\tilde{\Gamma}$ are δ -quasi-isomorphic if they are δ -close with δ -quasi-unitarily equivalent J , J' , J^1 and J'^1 resp. δ -quasi-isomorphic I and I' .

The δ -quasi-isomorphy only refers to the Dirichlet-to-Neumann form \mathfrak{l}_{-1} in $z = -1$ as $\|\varphi\|_{\mathcal{G}^{1/2}}^2 = \mathfrak{l}(\varphi) = \|S(-1)\varphi\|_{\mathcal{H}^1}^2$ and no other structure of Π ; a similar note holds for $\tilde{\Pi}$. We do not assume that I^* is closed to I' , as this is too restrictive for Definition 2.9 (see e.g. the proof of Proposition 2.11: $I^* = I'$ would mean $\gamma = 1$).

Finally, we define what it means for abstract boundary value problems to be “close” to each other, by combining the last four definitions:

2.9 Definition. Let $\delta > 0$. We say that the abstract boundary value problems Π and $\tilde{\Pi}$ are δ -quasi-isomorphic if there exist δ -quasi-unitarily equivalent identification operators J , J' , J^1 and J'^1 and δ -quasi-isomorphic identification operators I and I' for which \mathfrak{h} and $\tilde{\mathfrak{h}}$, respectively, Γ and $\tilde{\Gamma}$ are δ -close.

Let us illustrate this concept in two examples.

2.10 Example. A good test for a reasonable definition of a “distance” is the case $\delta = 0$: if Π and $\tilde{\Pi}$ are 0-quasi-isomorphic then J is unitary with adjoint J' ; J^1 and J'^1 are restrictions of J and J^* , respectively. Moreover, J intertwines H and \tilde{H} in the sense that $J(H + 1)^{-1} = (\tilde{H} + 1)^{-1}J$; and I is a bi-continuous isomorphism with inverse I' , and Γ and $\tilde{\Gamma}$ are equivalent in the sense that $\tilde{\Gamma} = I\Gamma J'^1$. We call such abstract boundary value problems *isomorphic*.

Another rather trivial case is the following: it nevertheless plays an important role in the study of shrinking domains like an ε -homothetic vertex neighbourhood shrinking to a point in the limit $\varepsilon \rightarrow 0$ (i.e., we use the abstract boundary value problem $\tilde{\Pi} = \Pi_\varepsilon$ associated with a compact and connected manifold X of dimension $d \geq 2$ with boundary $Y = \partial X$ and metric $\varepsilon^2 g$ as in Example 2.4 (1); in this case, $\delta = O(\sqrt{\varepsilon})$, see [P12, Sec. 5.1.4] for details, also for the validity of (2.7)):

2.11 Proposition. *Assume that $\tilde{\Pi} = (\tilde{\Gamma}, \tilde{\mathcal{G}}, \tilde{\mathfrak{h}}, \tilde{\mathcal{H}}^1, \tilde{\mathcal{H}})$ is an abstract boundary value problem such that the corresponding Neumann operator \tilde{H} has 0 as simple and isolated eigenvalue in its spectrum. Assume also that there is $a \in (0, 1]$ such that*

$$\|\tilde{\Gamma}u\|_{\tilde{\mathcal{G}}}^2 \leq a\tilde{\mathfrak{h}}(u) + \frac{2}{a}\|u\|_{\tilde{\mathcal{H}}}^2 \quad (2.7)$$

holds for all $u \in \tilde{\mathcal{H}}^1$.

Moreover, let $\Pi = (\text{id}, \mathbb{C}, 0, \mathbb{C}, \mathbb{C})$ be a trivial abstract boundary value problem. Then $\tilde{\Pi}$ and Π are δ -quasi-isomorphic with δ depending only on parameters of $\tilde{\Pi}$ and a , see (2.8) for a precise definition.

Proof. Let Φ_0 be a normalised eigenvector associated with the eigenvalue 0 of \tilde{H} . As $0 \in \sigma(\tilde{H})$, we also have $0 \in \sigma(\tilde{\Lambda}(0))$ with eigenvector $\Psi_0 = \tilde{\Gamma}\Phi_0$ (see [P16, Thm. 4.7 (i)]). In particular, $\gamma := \|\tilde{\Gamma}\Phi_0\|_{\tilde{\mathcal{G}}}^{-2}$ is defined. For the identification operators, we set

$$Jf = f\Phi_0, \quad J^1f = Jf, \quad J'u = J^*u = \langle u, \Phi_0 \rangle_{\tilde{\mathcal{H}}}, \quad J'^1u = J^*u, \quad I\varphi = \varphi\Psi_0$$

and $I' = \gamma I^*$, where $I^*\psi = \langle \psi, \Psi_0 \rangle_{\tilde{\mathcal{G}}}$. The choice of γ implies that $I'I\varphi = \varphi$, and

$$\|\psi - II'\psi\|_{\tilde{\mathcal{G}}}^2 = \|\psi - \gamma\langle \psi, \Psi_0 \rangle_{\tilde{\mathcal{G}}}\Psi_0\|_{\tilde{\mathcal{G}}}^2 \leq \frac{1}{\mu_1}\tilde{\mathfrak{l}}_0(\psi)$$

as $\sqrt{\gamma}\Psi_0$ is a normalised eigenfunction of $\tilde{\Lambda}(0)$ corresponding to the eigenvalue 0, where $\mu_1 := d(\sigma(\tilde{\Lambda}(0)) \setminus \{0\}, 0)$ and $\tilde{\mathfrak{l}}_0$ is the associated quadratic form. As $\lambda \mapsto \tilde{\mathfrak{l}}_\lambda$ is monotonously decreasing (see [P16, Thm. 2.12(v)]), we have the estimate $\tilde{\mathfrak{l}}_0(\psi) \leq \tilde{\mathfrak{l}}_{-1}(\psi) =: \|\psi\|_{\tilde{\mathcal{G}}^{1/2}}^2$. In particular, I and I' are $(1/\sqrt{\mu_1})$ -quasi-isomorphic, see Definition 2.8.

For the δ -closeness of the forms resp. the boundary maps we have

$$\begin{aligned} \tilde{\mathfrak{h}}(J^1f, u) - \mathfrak{h}(f, J'^1u) &= 0, \\ (I\Gamma - \tilde{\Gamma}J^1)f &= If - \tilde{\Gamma}f\Phi_0 = f \cdot (\Psi_0 - \tilde{\Gamma}\Phi_0) = 0 \quad \text{and} \\ (I'\tilde{\Gamma} - \Gamma J'^1)u &= \langle \gamma\tilde{\Gamma}u, \Psi_0 \rangle_{\tilde{\mathcal{G}}} - \langle u, \Phi_0 \rangle_{\tilde{\mathcal{H}}} \\ &= \langle \gamma\tilde{\Gamma}u, \Psi_0 \rangle_{\tilde{\mathcal{G}}} - \gamma\langle \Psi_0, \Psi_0 \rangle_{\tilde{\mathcal{G}}}\langle u, \Phi_0 \rangle_{\tilde{\mathcal{H}}} \\ &= \gamma\langle \tilde{\Gamma}(u - \langle u, \Phi_0 \rangle_{\tilde{\mathcal{H}}}\Phi_0), \Psi_0 \rangle_{\tilde{\mathcal{G}}}. \end{aligned}$$

The latter inner product can be estimated in squared absolute value by

$$\begin{aligned} |(I'\tilde{\Gamma} - \Gamma J^1)u|^2 &\leq \gamma \|\tilde{\Gamma}(u - \langle u, \Phi_0 \rangle_{\mathcal{H}} \Phi_0)\|_{\mathcal{G}}^2 \\ &\leq \gamma \left(a\tilde{\mathfrak{h}}(u) + \frac{2}{a} \|u - \langle u, \Phi_0 \rangle_{\mathcal{H}} \Phi_0\|_{\mathcal{H}}^2 \right) \\ &\leq \gamma \left(a + \frac{2}{a\lambda_1} \right) \tilde{\mathfrak{h}}(u), \end{aligned}$$

using (2.7), where $\lambda_1 := d(\sigma(\tilde{H}) \setminus \{0\}, 0)$. Note that $\tilde{\mathfrak{h}}(u - \langle u, \Phi_0 \rangle_{\mathcal{H}} \Phi_0) = \tilde{\mathfrak{h}}(u)$ as $\tilde{H}\Phi_0 = 0$ and hence $\tilde{\mathfrak{h}}(w, \Phi_0) = \langle w, \tilde{H}\Phi_0 \rangle = 0$ for any $w \in \tilde{\mathcal{H}}^1$.

Finally, $J^*Jf = f$ and $\|u - JJ^*u\|_{\mathcal{H}}^2 = \|u - \langle u, \Phi_0 \rangle_{\mathcal{H}} \Phi_0\|_{\mathcal{H}}^2 \leq \frac{1}{\lambda_1} \tilde{\mathfrak{h}}(u)$. Therefore we can choose

$$\delta = \max \left\{ \frac{1}{\sqrt{\mu_1}}, \frac{1}{\|\tilde{\Gamma}\Phi_0\|_{\mathcal{G}}} \sqrt{a + \frac{2}{a\lambda_1}}, \frac{1}{\sqrt{\lambda_1}} \right\}. \quad (2.8) \quad \square$$

3 Abstract graph-like spaces

Let us first explain the philosophy briefly. In the below-mentioned different couplings of abstract boundary value problems according to a graph, we show that the Neumann operator is coupled, while the Dirichlet operator is always a direct sum of the building blocks, i.e., decoupled. Moreover, we give formulas how the coupled operators can be calculated from the building blocks. We also analyse how the coupled operators such as the Dirichlet-to-Neumann operator resemble discrete Laplacians on the underlying or related graphs, allowing a deeper understanding of the problem and relating it to problems of graph Laplacians.

In particular, the resolvent formula (2.3) gives an expression of a globally defined object, namely the coupled Neumann operator in terms of objects from the building blocks (see e.g. the formulas for H^D , $S(z)$ and $\Lambda(z)$ in Theorems 3.3 and 3.7). Hence the understanding of the nature how $\Lambda(z)$ is obtained from the building blocks is essential in understanding the global operator H .

3.1 Direct sum of abstract boundary value problems

Given a family $(\Pi_\alpha)_{\alpha \in A}$ of abstract boundary value problems, we define the *direct sum* via³

$$\bigoplus_{\alpha \in A} \Pi_\alpha := \left(\bigoplus_{\alpha \in A} \Gamma_\alpha, \bigoplus_{\alpha \in A} \mathcal{G}_\alpha, \bigoplus_{\alpha \in A} \mathfrak{h}_\alpha, \bigoplus_{\alpha \in A} \mathcal{H}_\alpha^1, \bigoplus_{\alpha \in A} \mathcal{H}_\alpha \right).$$

The direct sum is an abstract boundary value problem provided $\sup_{\alpha \in A} \|\Gamma_\alpha\| < \infty$. As the direct sum is not coupled, we also call them *decoupled* and write

$$\Pi^{\text{dec}} = \left(\Gamma^{\text{dec}}, \mathcal{G}^{\text{dec}}, \mathfrak{h}^{\text{dec}}, \mathcal{H}^{1,\text{dec}}, \mathcal{H}^{\text{dec}} \right) := \bigoplus_{\alpha \in A} \Pi_\alpha.$$

All derived objects such as the Dirichlet solution operator or the Dirichlet-to-Neumann operator are also direct sums of the correspondent objects.

³The direct sum of Hilbert spaces always refers to the Hilbert space closure of the algebraic direct sum in this note.

3.2 Vertex coupling

We now construct a new space from building blocks associated with each vertex. Let $G = (V, E, \partial)$ be a graph. For each *vertex* $v \in V$ we assume that there is an abstract boundary value problem $\Pi_v = (\Gamma_v, \mathcal{G}_v, \mathfrak{h}_v, \mathcal{H}_v^1, \mathcal{H}_v)$.

3.1 Definition. We say that the family of abstract boundary value problems $(\Pi_v)_{v \in V}$ allows a *vertex coupling*, if the following holds:

1. Assume that $\sup_{v \in V} \|\Gamma_v\| < \infty$.
2. Assume there is a Hilbert space \mathcal{G}_e and a bounded operator $\pi_{v,e}: \mathcal{G}_v \rightarrow \mathcal{G}_e$ for each edge $e \in E$. Let $\mathcal{G}_v^{\max} := \bigoplus_{e \in E_v} \mathcal{G}_e$ and $\iota_v: \mathcal{G}_v \rightarrow \mathcal{G}_v^{\max}$, $\iota_v \varphi_v = (\pi_{v,e} \varphi_v)_{e \in E_v}$. We assume that ι_v is an isometric embedding.
3. Assume that $\text{ran } \Gamma_{\partial_{-e},e} = \text{ran } \Gamma_{\partial_{+e},e} =: \mathcal{G}_e^{1/2}$ for all edges $e \in E$, where $\Gamma_{v,e} := \pi_{v,e} \Gamma_v: \mathcal{H}_v^1 \rightarrow \mathcal{G}_e$.

We set $\pi_v := \iota_v^*: \mathcal{G}_v^{\max} \rightarrow \mathcal{G}_v$, then $\pi_v \psi = \sum_{e \in E_v} \pi_{v,e}^* \psi_e$. We say that the vertex coupling is *maximal* if ι_v is surjective (hence unitary). In this case, we often identify \mathcal{G}_v with \mathcal{G}_v^{\max} .

We start with an example with maximal vertex coupling spaces $\mathcal{G}_v \cong \mathcal{G}_v^{\max}$ (an example with non-maximal vertex coupling spaces will be given in Section 3.5):

3.2 Example. Assume that we have a graph-like manifold X (without edge contributions), i.e., $X = \bigcup_{v \in V} X_v$ such that X_v is closed in X and $Y_e := X_{\partial_{-e}} \cap X_{\partial_{+e}}$ is a smooth submanifold. Then $\mathcal{H}_v = \mathbf{L}_2(X_v)$, $\mathfrak{h}_v(f) = \int_{X_v} |df|^2$, $\text{dom } \mathfrak{h}_v = \mathcal{H}_v^1 = \mathbf{H}^1(X_v)$ and $\mathcal{G}_v = \mathbf{L}_2(\partial X_v)$, $\Gamma_v f = f|_{\partial X_v}$; and $\Pi_v = (\Gamma_v, \mathcal{G}_v, \mathfrak{h}_v, \mathcal{H}_v^1, \mathcal{H}_v)$ is an abstract boundary value problem. Moreover, $(\Pi_v)_v$ allows a vertex coupling with $\mathcal{G}_e = \mathbf{L}_2(Y_e)$ with maps $\pi_{v,e}: \mathbf{L}_2(\partial X_v) \rightarrow \mathbf{L}_2(Y_e)$ being the restriction of a function on ∂X_v onto one of its components $Y_e \subset \partial X_v$. Note that $\text{ran } \Gamma_{\partial_{\pm e},e} = \mathbf{H}^{1/2}(Y_e)$. As

$$\mathcal{G}_v^{\max} := \bigoplus_{e \in E_v} \mathcal{G}_e = \bigoplus_{e \in E_v} \mathbf{L}_2(Y_e) = \mathbf{L}_2(\partial X_v) = \mathcal{G}_v,$$

the vertex coupling is *maximal*. Condition (1) is typically fulfilled, if the length of each end of a building block X_v is bounded from below by some constant $\ell_0/2 > 0$.

We construct an abstract boundary value problem $\Pi = (\Gamma, \mathcal{G}, \mathfrak{h}, \mathcal{H}^1, \mathcal{H})$ from $(\Pi_v)_v$ as follows:

$$\begin{aligned} \mathcal{H} &:= \bigoplus_{v \in V} \mathcal{H}_v, & \mathcal{H}^{1,\text{dec}} &:= \bigoplus_{v \in V} \mathcal{H}_v^1, & \mathfrak{h}^{\text{dec}} &:= \bigoplus_{v \in V} \mathfrak{h}_v; \\ \mathcal{H}^1 &:= \left\{ f = (f_v)_{v \in V} \in \mathcal{H}^{1,\text{dec}} \mid \forall e \in E, \partial e = (v, w): \Gamma_{v,e} f_v = \Gamma_{w,e} f_w =: \Gamma_e f \right\}, \\ \mathfrak{h} &:= \mathfrak{h}^{\text{dec}}|_{\mathcal{H}^1}, & \mathcal{G} &:= \bigoplus_{e \in E} \mathcal{G}_e, & \Gamma: \mathcal{H}^1 &\rightarrow \mathcal{G}, \quad \Gamma f = (\Gamma_{\partial_{\pm e},e} f_{\partial_{\pm e}})_{e \in E}. \end{aligned}$$

Denote by ι the map

$$\iota: \mathcal{G} = \bigoplus_{e \in E} \mathcal{G}_e \rightarrow \mathcal{G}^{\text{dec}} = \bigoplus_{v \in V} \mathcal{G}_v, \quad (\iota \varphi)_v = \pi_v \varphi(v) = \sum_{e \in E_v} \pi_{v,e}^* \varphi_e,$$

where $\varphi(v) = (\varphi_e)_{e \in E_v} \in \mathcal{G}_v^{\max}$ (see Definition 3.1 for the notation). It is easy to see that $\iota^*: \mathcal{G}^{\text{dec}} \rightarrow \mathcal{G}$ acts as

$$(\iota^* \psi)_e = \sum_{v=\partial_{\pm e}} \pi_{v,e} \psi(v)$$

3.3 Theorem. Assume that $(\Pi_v)_{v \in V}$ is a family of abstract boundary value problems allowing a vertex coupling, then the following holds:

1. The quintuple $\Pi = (\Gamma, \mathcal{G}, \mathfrak{h}, \mathcal{H}^1, \mathcal{H})$ as constructed above is an abstract boundary value problem.
2. We have $\ker \Gamma = \bigoplus_{v \in V} \ker \Gamma_v$, $H^D = \bigoplus_{v \in V} H_v^D$ (i.e., the Dirichlet operator is decoupled) and $\sigma(H^D) = \bigcup_{v \in V} \sigma(H_v^D)$.
3. In particular, if all abstract boundary value problems Π_v have dense Dirichlet domain then Π also has dense Dirichlet domain.
4. The Neumann operator is coupled and $f \in \text{dom } H$ iff $f \in \bigoplus_{v \in V} \mathcal{W}_v$ and

$$\Gamma_{v,e} f_v = \Gamma_{w,e} f_w, \quad \Gamma'_{v,e} f_v + \Gamma'_{w,e} f_w = 0 \quad \forall e \in E, \partial e = (v, w)$$

with $Hf = (H^{\max} f)_{v \in V}$ (see (2.2) for the notation).

5. We have $S(z)\varphi = S^{\text{dec}}(z)\iota\varphi = \bigoplus_{v \in V} S_v(z)\varphi(v)$ for $z \notin \sigma(H^D)$.
6. Moreover, if all Π_v are elliptically regular such that $\sup_{v \in V} \|S_v\|_{\mathcal{G}_v \rightarrow \mathcal{H}_v} < \infty$, then Π is also elliptically regular.
7. We have $\Lambda(z) = \iota^* \Lambda^{\text{dec}}(z)\iota$ for $z \in \mathbb{C} \setminus \sigma(H^D)$, i.e.,

$$(\Lambda(z)\varphi)_e = \sum_{v=\partial_{\pm}e} \pi_{v,e}(\Lambda_v(z)\varphi(v)).$$

Proof. (1) The space \mathcal{H}^1 is closed in $\mathcal{H}^{1,\text{dec}}$ as intersection of the closed spaces

$$\{ f \in \mathcal{H}^{1,\text{dec}} \mid \Gamma_{\partial_{-}e,e} f_{\partial_{-}e} = \Gamma_{\partial_{+}e,e} f_{\partial_{+}e} \}$$

(note that since $\Gamma_{v,e}$ are bounded operators, the latter sets are closed). Hence \mathcal{H}^1 is closed and \mathfrak{h} is a closed form. Moreover, the operator Γ is bounded, as

$$\|\Gamma f\|_{\mathcal{G}}^2 = \sum_{e \in E} \|\Gamma_e f\|_{\mathcal{G}_e}^2 = \frac{1}{2} \sum_{v \in V} \sum_{e \in E_v} \|\Gamma_{v,e} f_v\|_{\mathcal{G}_e}^2 = \frac{1}{2} \sum_{v \in V} \|\iota_v \Gamma_v f_v\|_{\mathcal{G}_v^{\max}}^2$$

using (2.1) and this can be estimated by $\sup_v \|\Gamma_v\|^2 \|f\|_{\mathcal{H}^1}^2 / 2$ as ι_v is isometric.

Finally, as $\text{ran } \Gamma_v = \mathcal{G}_v^{1/2}$ is dense in \mathcal{G}_v we also have that $\text{ran } \Gamma_{v,e}$ is dense in $\mathcal{G}_{v,e}$ (applying the bounded operator $\pi_{v,e}: \mathcal{G}_v \rightarrow \mathcal{G}_e$ to a dense set). As $\text{ran } \Gamma = \bigoplus_{e \in E} \text{ran } \Gamma_e$ (algebraic direct sum) with $\text{ran } \Gamma_e = \text{ran } \Gamma_{\partial_{\pm}e,e}$, the density of $\text{ran } \Gamma$ in \mathcal{G} follows.

(2) We have $f \in \bigoplus_{v \in V} \ker \Gamma_v$ iff $f_v \in \ker \Gamma_v$ for all $v \in V$. Moreover, $\ker \Gamma_v = \bigcap_{e \in E_v} \ker \Gamma_{v,e}$ as $\bigcap_{e \in E_v} \ker \pi_{v,e} = \{0\} \subset \mathcal{G}_v$ (using the injectivity of ι_v , see Definition 3.1 (2)). By definition, $\Gamma_e := \Gamma_{v,e}$ for $v = \partial_{\pm}e$, hence we have

$$\ker \Gamma = \bigoplus_{e \in E} \ker \Gamma_e = \bigoplus_{v \in V} \ker \Gamma_v.$$

(3) In particular, if all spaces $\ker \Gamma_v$ are dense in \mathcal{H}_v , then $\ker \Gamma$ is dense in \mathcal{H} .

(4) follows from a simple calculation using (2.2) on each abstract boundary value problem Π_v .

(5) is obvious, as well as (6)

(7) The formula follows from $\mathfrak{L}_z(\varphi, \psi) = (\mathfrak{h} - z\mathbf{1})(S(z)\varphi, S(-1)\psi)$ and part (5). \square

3.4 Examples.

1. In Example 3.2, the entire space is $\mathcal{H} = L_2(X)$ and the Neumann operator is the usual Laplacian on the graph-like manifold X .
2. Assume that $G = (V, E, \partial)$ is a discrete graph. We decompose G into its *star components* $G_v = (\{v\} \cup E_v, E_v, \tilde{\partial})$, i.e., each edge in G adjacent to v becomes also a vertex in G_v . As boundary of G_v we set $\partial G_v = E_v$. If we identify the new vertices $e \in V(G_{\partial_- e})$ and $e \in V(G_{\partial_+ e})$ of the star components $G_{\partial_- e}$ and $G_{\partial_+ e}$ for all edges $e \in E$ we just obtain the subdivision graph SG (see Definition 2.1).

Let Π_v be the abstract boundary value problem associated with the graph G_v and boundary $\partial G_v = E_v$, i.e., $\Pi_v = (\mathbb{C}^{E_v}, \Gamma_v, \mathfrak{h}_v, \mathbb{C}^{E_v \cup \{v\}}, \mathbb{C}^{E_v \cup \{v\}})$ with $\Gamma_v f = f|_{E_v}$ (see Example 2.4 (3)), where $\mathbb{C}^{E_v} := \{ \varphi: E_v \rightarrow \mathbb{C} \mid \varphi \text{ map} \}$ denotes the set of maps or families with coordinates indexed by $e \in E_v$. Denote for short $d_v = \deg v$. Of course, \mathbb{C}^{E_v} is isomorphic to \mathbb{C}^{d_v} , but this isomorphism needs a numbering of the edges which is unimportant for our purposes. The Neumann operator, written as a matrix with respect to the orthonormal basis $\varphi_v := d_v^{-1/2} \delta_v$ ($v \in V$), has block structure $A_v = \text{id}_{E_v}$ (identity matrix of dimension d_v), $B_v = -d_v^{-1/2}(1, \dots, 1)^T$ and $D_v = 1$. In particular, the Dirichlet-to-Neumann operator is $\Lambda_v(z) = (1 - z) \text{id}_{E_v} - (d_v(1 - z))^{-1} \mathbb{1}_{E_v \times E_v}$, where $\mathbb{1}_{E_v \times E_v}$ is the $(d_v \times d_v)$ -matrix with all entries 1.

The vertex-coupled abstract boundary value problem Π of the family $(\Pi_v)_{v \in V}$ (it is clear that this family allows a vertex coupling) is now the abstract boundary value problem of the subdivision graph SG of G with boundary $\partial SG = E$, the edges of G . More precisely, we have already identified the subspace $\{ \varphi \in \bigoplus_{v \in V} \ell_2(G_v) \mid f_{v,e} = f_{w,e} \ \forall e \in E, \partial e = (v, w) \}$, with $\ell_2(SG)$. The coupled Neumann operator is the Laplacian of the subdivision graph SG , i.e., $H = \Delta_{SG}$. Note that we can embed $\ell_2(G)$ into $\ell_2(SG)$, $f \mapsto \tilde{f}$, with $\tilde{f}(v) = f(v)$ and $\tilde{f}(e) = (f(\partial_+ e) + f(\partial_- e))/2$; moreover, $2\mathfrak{h}_G(f) = \mathfrak{h}_{SG}(\tilde{f})$ for the corresponding energy forms. The coupled Dirichlet-to-Neumann operator is

$$(\Lambda(z)\varphi)_e = 2(1 - z)\varphi_e - \frac{1}{1 - z} \sum_{v=\partial_{\pm} e} \frac{1}{\deg v} \sum_{e' \in E_v} \varphi_{e'}.$$

For $z = 0$, this is just the formula for a Laplacian on the line graph LG of G (the line graph has as vertices the edges of G , and two such edges are adjacent, if they meet in a common vertex, see e.g. [P09, Ex. 3.14 (iv)]). In particular, if G is r -regular, then LG is $(2r - 2)$ -regular and

$$\Lambda(z) = 2(1 - z) - \frac{2r - 2}{(1 - z)r} (1 - \Delta_{LG}). \quad (3.1)$$

In particular, applying the spectral relation (2.4) to the last example (the Dirichlet spectrum of all star components is $\{1\}$ as $H_v^D = D_v = 1$) we rediscover the following result (see [Sh00]):

3.5 Corollary. *The spectra of the subdivision and line graph of an r -regular graph are related by*

$$\lambda \in \sigma(\Delta_{SG}) \iff 1 - \frac{r}{r - 1} (1 - \lambda)^2 \in \sigma(\Delta_{LG})$$

provided $\lambda \neq 1$.

3.3 Edge coupling

Let us now couple abstract boundary value problems indexed by the edges of a given graph $G = (V, E, \partial)$: For each edge $e \in E$ we assume that there is an abstract boundary value problem $\Pi_e = (\Gamma_e, \mathcal{G}_e, \mathfrak{h}_e, \mathcal{H}_e^1, \mathcal{H}_e)$.

3.6 Definition. We say that the family of abstract boundary value problems $(\Pi_e)_{e \in E}$ allows an *edge coupling*, if the following holds:

1. Assume that $\sup_{e \in E} \|\Gamma_e\| < \infty$.
2. Assume that for each vertex $v \in V$ there is a decomposition $\mathcal{G}_e = \mathcal{G}_{e, \partial_- e} \oplus \mathcal{G}_{e, \partial_+ e}$ and $\Gamma_e f = \Gamma_{e, \partial_- e} f \oplus \Gamma_{e, \partial_+ e} f$, where $\Gamma_{e, v}: \mathcal{H}_e^1 \rightarrow \mathcal{G}_{e, v}$.

We set $\mathcal{G}_v^{\max} := \bigoplus_{e \in E_v} \mathcal{G}_{e, v}$.

Note that the sum over all maximal spaces \mathcal{G}_v^{\max} is the decoupled space, as

$$\mathcal{G}^{\max} := \bigoplus_{v \in V} \mathcal{G}_v^{\max} = \bigoplus_{v \in V} \bigoplus_{e \in E_v} \mathcal{G}_{e, v} = \bigoplus_{e \in E} \bigoplus_{v = \partial_{\pm} e} \mathcal{G}_{e, v} = \bigoplus_{e \in E} \mathcal{G}_e = \mathcal{G}^{\text{dec}}. \quad (3.2)$$

Denote $\underline{\varphi}(v) := (\varphi_e(v))_{e \in E_v} \in \mathcal{G}_v^{\max}$ the collection of all edge contributions at the vertex $v \in V$, where $\varphi_e = (\varphi_e(\partial_- e), \varphi_e(\partial_+ e)) \in \mathcal{G}_{e, \partial_- e} \oplus \mathcal{G}_{e, \partial_+ e}$.

Let $\mathcal{G}_v \subset \mathcal{G}_v^{\max}$ be a closed subspace for each $v \in V$. We construct an abstract boundary value problem $\Pi = (\Gamma, \mathcal{G}, \mathfrak{h}, \mathcal{H}^1, \mathcal{H})$ from the family $(\Pi_e)_e$ and the subspace $\mathcal{G} := \bigoplus_v \mathcal{G}_v$ as a restriction of the decoupled abstract boundary value problem $\bigoplus_{e \in E} \Pi_e$ (see Section 3.1):

$$\begin{aligned} \mathcal{H} &:= \bigoplus_{e \in E} \mathcal{H}_e, & \mathcal{H}^{1, \text{dec}} &:= \bigoplus_{e \in E} \mathcal{H}_e^1, & \mathfrak{h}^{\text{dec}} &:= \bigoplus \mathfrak{h}_e; \\ \mathcal{H}^1 &:= \{ f = (f_e)_{e \in E} \in \mathcal{H}^{1, \text{dec}} \mid \Gamma^{\text{dec}} f \in \mathcal{G} \}, & \mathfrak{h} &:= \mathfrak{h}^{\text{dec}}|_{\mathcal{H}^1}, & \Gamma &:= \Gamma^{\text{dec}}|_{\mathcal{H}^1} \end{aligned}$$

where $\Gamma^{\text{dec}}: \mathcal{H}^{1, \text{dec}} \rightarrow \mathcal{G}^{\text{dec}} = \mathcal{G}^{\max}$. Denote by ι the embedding $\iota: \mathcal{G} \rightarrow \mathcal{G}^{\text{dec}}$.

3.7 Theorem. Assume that $(\Pi_e)_{e \in E}$ is a family of abstract boundary value problems allowing an edge coupling and let $\mathcal{G}_v \subset \mathcal{G}_v^{\max}$ be a closed subspace for each $v \in V$, then the following holds:

1. The quintuple $\Pi = (\Gamma, \mathcal{G}, \mathfrak{h}, \mathcal{H}^1, \mathcal{H})$ as constructed above is an abstract boundary value problem.
2. We have $\ker \Gamma = \bigoplus_{e \in E} \ker \Gamma_e$, $H^{\text{D}} = \bigoplus_{e \in E} H_e^{\text{D}}$ (the Dirichlet operator is decoupled) and $\sigma(H^{\text{D}}) = \bigcup_{e \in E} \sigma(H_e^{\text{D}})$.
3. In particular, if all abstract boundary value problems Π_e have dense Dirichlet domain then Π also has dense Dirichlet domain.
4. The Neumann operator is coupled and is given by

$$\text{dom } H = \left\{ f \in \bigoplus_{e \in E} \mathcal{W}_e \mid \Gamma f \in \mathcal{G}, \Gamma' f \in \mathcal{G}^{\max} \ominus \mathcal{G} \right\}$$

with $Hf = (H^{\max} f)_{v \in V}$ (see (2.2) for the notation).

5. We have $S(z)\varphi = S^{\text{dec}}(z)\varphi = \bigoplus_{e \in E} S_e(z)\varphi_e$ where $\varphi \in \mathcal{G} \subset \mathcal{G}^{\text{dec}}$.

6. Moreover, if all Π_e are elliptically regular with uniformly bounded elliptic regularity constants (i.e., $\sup_{e \in E} \|S_e\|_{\mathcal{G}_e \rightarrow \mathcal{H}_e} < \infty$), then Π is also elliptically regular.
7. We have $\Lambda(z) = \iota^* \Lambda^{\text{dec}}(z) \iota$. i.e., if $\psi = \Lambda(z)\varphi$, then

$$\underline{\psi}(v) := (\psi_e)_{e \in E_v} = \pi_v((\Lambda_e(z)\varphi_e)(v))_{e \in E_v}$$

for $z \notin \sigma(H^D)$, where $\pi_v: \mathcal{G}_v^{\text{max}} \rightarrow \mathcal{G}_v$ is the adjoint of $\iota_v: \mathcal{G}_v \rightarrow \mathcal{G}_v^{\text{max}}$.

Proof. The proof is very much as the proof of Theorem 3.3: (1) The operator Γ^{dec} is bounded, and \mathcal{H}^1 is closed in $\mathcal{H}^{1,\text{dec}}$ as preimage of the closed subspace \mathcal{G} under Γ^{dec} ; in particular, Γ is bounded. Moreover, $\text{ran } \Gamma = \Gamma(\mathcal{H}^1) = \Gamma^{\text{dec}}(\mathcal{H}^1) = \mathcal{G} \cap \Gamma^{\text{dec}}(\mathcal{H}^1)$; since $\Gamma^{\text{dec}}(\mathcal{H}^1)$ is dense (as all components $\Gamma_e(\mathcal{H}_e^1)$ are dense in \mathcal{G}_e , the space $\text{ran } \Gamma$ is also dense in \mathcal{G} .

(2)–(7) can be seen similarly as in the proof of Theorem 3.3. \square

3.8 Examples. Let us give some important examples of subspaces \mathcal{G} of \mathcal{G}^{max} :

1. **Edge coupling of two-dimensional abstract boundary value problems:** Assume that all vertex components $\mathcal{G}_{e,v}$ equal \mathbb{C} . Then $\mathcal{G}_v^{\text{max}} = \mathbb{C}^{E_v}$ and we choose for example $\mathcal{G}_v := \mathbb{C}(1, \dots, 1)$ (standard or Kirchhoff vertex conditions), where $(1, \dots, 1) \in \mathbb{C}^{E_v}$ has all $(\deg v)$ -many components 1. It is convenient to choose $|w|_{\deg v} = |w| \sqrt{\deg v}$ as norm on \mathbb{C} (then $\mathbb{C}(1, \dots, 1) \subset \mathbb{C}^{E_v}$ is isometrically embedded in $(\mathbb{C}, |\cdot|_{\deg v})$ via $(\eta, \dots, \eta) \mapsto \eta$). A vector $\underline{\eta}(v)$ is of course characterised by the common scalar value $\eta(v) \in \mathbb{C}$ and the projection $\pi_v \underline{\psi}(v) = (\psi_e(v))_{e \in E_v}$ is characterised by the sum $(\deg v)^{-1} \sum_{e \in E_v} \psi_e(v)$. Hence we can write the Dirichlet-to-Neumann operator as

$$(\Lambda(z)\varphi)(v) = \frac{1}{\deg v} \sum_{e \in E_v} (\Lambda_e(z)\varphi_e)(v) \quad (3.3)$$

for $\varphi \in \mathcal{G} = \ell_2(V, \deg)$. This formula is a generalisation of the formula for the discrete (normalised) Laplacian. One can also choose a more general subspace $\mathcal{G}_v \subset \mathcal{G}_v^{\text{max}} = \mathbb{C}^{E_v}$, the resulting Dirichlet-to-Neumann operators look like generalised discrete Laplacians described e.g. in [P08, Sec. 3] or [P09, Sec. 3]. A similar approach has been used in [Pa06].

2. **Edge coupling of two-dimensional *trivial* abstract boundary value problems gives back the original discrete graph:** Let us now treat a special case of (1): Let $\Pi_e = (\text{id}, \mathbb{C}^2, \mathfrak{h}_e, \mathbb{C}^2, \mathbb{C}^2)$ be a trivial abstract boundary value problem for each $e \in E$. The abstract boundary value problem Π_v can be understood as coming from a graph consisting only of two vertices $\partial_{\pm} e$ and one edge e , and both vertices belong to the boundary. The energy form is $\mathfrak{h}_e(f) = |f_2 - f_1|^2$ for $f = (f_1, f_2) \in \mathbb{C}^2$, see Example 2.4 (3)). In this case,

$$\Lambda_e(z) = \begin{pmatrix} 1-z & -1 \\ -1 & 1-z \end{pmatrix}$$

and the Dirichlet-to-Neumann operator becomes

$$(\Lambda(z)\varphi)(v) = \frac{1}{\deg v} \sum_{e \in E_v} (\varphi(v) - \varphi(v_e)) - z\varphi(v),$$

i.e., $\Lambda(z) = \Delta_G - z$, i.e., the Dirichlet-to-Neumann operator is just the shifted normalised Laplacian on G . Note that in this case, the Neumann Laplacian is also the Dirichlet-to-Neumann operator at $z = 0$, i.e., $H = \Lambda(0) = \Delta_G$, as the global form \mathfrak{h} is $\mathfrak{h}(f) = \sum_e \mathfrak{h}_e(f_e)$.

3. Standard vertex conditions: Here, we describe an edge coupling with a special choice of vertex spaces $\mathcal{G}_v \subset \mathcal{G}_v^{\max}$, similar to the standard or Kirchhoff vertex conditions of a quantum graph.

Assume that for given $v \in V$, the vertex component $\mathcal{G}_{e,v}$ of \mathcal{G}_e equals a given Hilbert space $\mathcal{G}_{v,0}$ for all $e \in E_v$. Then $\mathcal{G}_v := \{ \underline{\eta}(v) = (\eta, \dots, \eta) \in \mathcal{G}_v^{\max} \mid \eta \in \mathcal{G}_{v,0} \}$ is a closed subspace (i.e., \mathcal{G}_v consists of the $(\deg v)$ -fold diagonal of the model space $\mathcal{G}_{v,0}$). In the above two examples, we treated the case $\mathcal{G}_{e,v} = \mathbb{C}$. A vector $\underline{\eta}(v)$ is characterised by $\eta(v)$ and the projection $\pi_v \underline{\psi}(v) = (\psi_e(v))_{e \in E_v}$ is characterised by the sum $(\deg v)^{-1} \sum_{e \in E_v} \psi_e(v)$. Hence we can write the Dirichlet-to-Neumann operator exactly as in (3.3). This formula is a vector-valued version of a normalised discrete Laplacian (see (3.7) for a more concrete formula).

3.4 Vector-valued quantum graphs

Let $I = [a, b]$ be a compact interval of length $\ell = b - a > 0$, and let \mathcal{K} be a Hilbert space with non-negative closed quadratic form $\mathfrak{k} \geq 0$ such that its corresponding operator has purely discrete spectrum. We set $\mathcal{H} := \mathbf{L}_2(I, \mathcal{K}) \cong \mathbf{L}_2(I) \otimes \mathcal{K}$ and define an energy form via

$$\mathfrak{h}(f) := \int_I (\|f'(t)\|_{\mathcal{K}}^2 + \mathfrak{k}(f(t))) \, dt$$

for $f \in \mathbf{L}_2(I, \mathcal{K})$ such that $f \in \mathbf{C}^1(I, \mathcal{K})$ and $f(t) \in \text{dom } \mathfrak{k}$ for almost all $t \in I$. Denote by \mathfrak{h} also the closure of this form. As boundary space set $\mathcal{G} := \mathcal{K} \oplus \mathcal{K} \cong \mathcal{K} \otimes \mathbb{C}^2$ and define $\Gamma f := (f(a), f(b))$. It is not hard to see that $\Pi = (\Gamma, \mathcal{G}, \mathfrak{h}, \text{dom } \mathfrak{h}, \mathcal{H})$ is an elliptically regular abstract boundary value problem with dense Dirichlet domain; moreover, the norm of Γ is bounded by $\sqrt{\coth(\ell/2)}$ (see e.g. [P16, Sec. 6.1 and 6.4]). We call Π the *abstract boundary value problem associated with* $(\mathfrak{k}, \mathcal{K}, I)$. Moreover, as the underlying space of Π has a product structure, we can calculate all derived objects explicitly. For example, the Dirichlet-to-Neumann operator $\Lambda(z)$ is an operator function of a matrix with respect to the decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{K}$. In particular, we have $\Lambda(z) = \Lambda_0(z - K)$, where K is the operator associated with \mathfrak{k} and where

$$\Lambda_0(z) = \frac{\sqrt{z}}{\sin(\ell\sqrt{z})} \begin{pmatrix} \cos(\ell\sqrt{z}) & -1 \\ -1 & \cos(\ell\sqrt{z}) \end{pmatrix} \quad (3.4)$$

is the Dirichlet-to-Neumann operator for the scalar problem ($\mathcal{K} = \mathbb{C}$)

$$\Pi_0 = (\Gamma_0, \mathbb{C}^2, \mathfrak{h}_0, \mathbf{H}^1(I), \mathbf{L}_2(I))$$

with $\Gamma_0 f = (f(a), f(b))$ and $\mathfrak{h}_0(f) = \int_I |f'(t)|^2 \, dt$ (see e.g. [BP16] for details). The complex square root is cut along the positive real axis. The same argument also works for abstract warped products.

Let us now consider vector-valued quantum graphs: Assume that $G = (V, E, \partial)$ is a discrete graph and that I_e is a closed interval of length ℓ_e for each $e \in E$. Assume that there is a Hilbert space \mathcal{K}_e and energy form \mathfrak{k}_e for each edge $e \in E$. Then we can edge-couple the family of abstract boundary value problem Π_e associated with $(\mathfrak{k}_e, \mathcal{K}_e, I_e)$. In order to formulate the next result, we define the *unoriented* and *oriented evaluation* of f at a vertex v and edge e by

$$f_e(v) = \begin{cases} f_e(\min I_e), & v = \partial_- e \\ f_e(\max I_e), & v = \partial_+ e \end{cases} \quad \text{and} \quad \widehat{f}_e(v) = \begin{cases} -f_e(\min I_e), & v = \partial_- e \\ +f_e(\max I_e), & v = \partial_+ e \end{cases}$$

for $f \in \bigoplus_{e \in E} \mathbf{H}^1(I_e, \mathcal{K}_e)$.

3.9 Theorem. *Let $\Pi_e = (\Gamma_e, \mathcal{K}_e \oplus \mathcal{K}_e, \mathfrak{h}_e, \text{dom } \mathfrak{h}_e, \mathbf{L}_2(I_e, \mathcal{K}_e))$ be an abstract boundary value problem associated with $(\mathfrak{k}_e, \mathcal{K}_e, I_e)$. Assume that the length ℓ_e of I_e fulfils*

$$0 < \ell_0 := \inf_{e \in E} \ell_e. \quad (3.5)$$

1. *Then the family $(\Pi_e)_{e \in E}$ allows an edge coupling. As boundary space we choose $\mathcal{G} = \bigoplus_{v \in V} \mathcal{G}_v$, where \mathcal{G}_v is a closed subspace of $\mathcal{G}_v^{\max} := \bigoplus_{e \in E_v} \mathcal{K}_e$ for each $v \in V$. Then H acts as*

$$(Hf)_e(t) = -f_e''(t) + K_e f(t) \quad (3.6a)$$

on each edge, where K_e is the operator associated with \mathfrak{k}_e . Moreover, a function f in the domain of the Neumann operator H of the edge-coupled abstract boundary value problem fulfils

$$\underline{f}(v) = (f_e(v))_{e \in E_v} \in \mathcal{G}_v \quad \text{and} \quad \underline{\hat{f}}'(v) = (\hat{f}_e'(v))_{e \in E_v} \in \mathcal{G}_v^{\max} \ominus \mathcal{G}_v. \quad (3.6b)$$

2. *If H is a self-adjoint operator in $\mathcal{H} = \bigoplus_{e \in E} \mathbf{L}_2(I_e, \mathcal{K}_e)$ such that (3.6a) holds for all functions $f = (f_e)_e \in \text{dom } H$ such that $f_e \in \mathbf{C}^2(I_e, \text{dom } K_e)$ vanishing near ∂I_e , and such that the values $\underline{f}(v)$ and $\underline{\hat{f}}'(v)$ are not coupled in $\text{dom } H$, then there exist closed subspaces \mathcal{G}_v of \mathcal{G}_v^{\max} for each $v \in V$, such that $\text{dom } H$ is of the form (3.6b)*

We call H the *vector-valued quantum graph Laplacian with vertex spaces \mathcal{G}_v and fibre operators K_e* .

Proof. Part (1) follows already from the discussion above, the fact that $\Gamma f = (\underline{f}(v))_{v \in V}$ and $\Gamma' f = (\underline{\hat{f}}'(v))_{v \in V}$, and Theorem 3.7. Note that $\|\Gamma_e\|^2$ is bounded by $2/\min\{\ell_0, 1\}$, see [P12, Cor. A.2.12].

For part (2), partial integration shows that

$$\begin{aligned} \langle Hf, g \rangle_{\mathcal{H}} &= \sum_{e \in E} \int_{I_e} \langle -f_e''(t) + K_e f_e(t), g_e(t) \rangle_{\mathcal{K}_e} dt \\ &= \sum_{e \in E} \left(\int_{I_e} (\langle f_e'(t), g_e'(t) \rangle_{\mathcal{K}_e} + \mathfrak{k}_e(f_e(t), g_e(t))) dt - \left[\langle f_e'(t), g_e(t) \rangle_{\mathcal{K}_e} \right]_{\partial I_e} \right) \\ &= \langle f, Hg \rangle_{\mathcal{H}} + \sum_{e \in E} \left[\langle f_e(t), g_e'(t) \rangle_{\mathcal{K}_e} - \langle f_e'(t), g_e(t) \rangle_{\mathcal{K}_e} \right]_{\partial I_e} \end{aligned}$$

for $f, g \in \text{dom } H$. Reordering the boundary contributions (the last sum over $e \in E$) gives

$$\begin{aligned} &\sum_{v \in V} \sum_{e \in E_v} (\langle \hat{f}_e'(v), g_e(v) \rangle_{\mathcal{K}_e} - \langle f_e(v), \hat{g}_e'(v) \rangle_{\mathcal{K}_e}) \\ &= \sum_{v \in V} (\langle \underline{\hat{f}}'(v), \underline{g}(v) \rangle_{\mathcal{G}_v^{\max}} - \langle \underline{f}(v), \underline{\hat{g}}'(v) \rangle_{\mathcal{G}_v^{\max}}), \end{aligned}$$

As the values $\underline{\hat{f}}'(v)$ and $\underline{f}(v)$ resp. $\underline{\hat{g}}'(v)$ and $\underline{g}(v)$ are not coupled, each contribution $\langle \underline{\hat{f}}'(v), \underline{g}(v) \rangle_{\mathcal{G}_v^{\max}}$ and $\langle \underline{f}(v), \underline{\hat{g}}'(v) \rangle_{\mathcal{G}_v^{\max}}$ has to vanish separately. We let \mathcal{G}_v be the closure of the linear span of all boundary values $\underline{f}(v)$, $f \in \text{dom } H$. In particular, we then have $\underline{\hat{f}}'(v), \underline{\hat{g}}'(v) \in \mathcal{G}_v^\perp$. \square

3.10 Remarks.

1. For simplicity, we describe only the *energy independent* vertex conditions, not involving any condition between the values $\underline{f}(v)$ and $\underline{f}'(v)$. One can, of course, also consider Robin-type conditions, but one needs additional finiteness or boundedness conditions in this case.
2. If $\mathcal{K}_e = \mathbb{C}$ for all edges $e \in E$, then we have defined an ordinary quantum graph. For the case $(\mathfrak{k}_e, \mathcal{K}_e, I_e) = (0, \mathcal{K}_0, [0, 1])$ for all $e \in E$, where \mathcal{K}_0 is a given Hilbert space, see [vBM13].

We have not used the whole power of abstract boundary value problems here, namely the resolvent formula (2.3). In this setting, the left hand side, the resolvent of H in $z \notin \sigma(H) \cup \sigma(H^D)$, equals the right hand side, which can be expressed completely in terms of the building blocks Π_e . Moreover, the Dirichlet-to-Neumann operator has the nature of a discrete Laplacian.

If we use standard vertex conditions (see Example 3.8 (3)), we have to assume that all boundary spaces \mathcal{K}_e are the same and equal (or can at least be naturally identified with) a Hilbert space \mathcal{K}_0 . Then we set $\mathcal{G}_v = \mathcal{K}_0(1, \dots, 1)$, i.e., all $\deg v$ components of $\underline{\varphi}(v) \in \mathcal{G}_v$ are the same. In this case, the Dirichlet-to-Neumann operator is (see (3.3) and (3.4))

$$(\Lambda(z)\varphi)(v) = \frac{1}{\deg v} \sum_{e \in E_v} (C_e(z)\varphi(v) - S_e(z)\varphi(v_e)) \quad (3.7)$$

where $C_e(z) = \sqrt{z - K_e} \cot(\ell_e \sqrt{z - K_e})$ and $S_e(z) = \sqrt{z - K_e} / \sin(\ell_e \sqrt{z - K_e})$.

The equilateral and standard (or Kirchhoff) case. Let us now characterise the spectrum of the vector-valued quantum graph Laplacian H in a special case:

3.11 Example. If all abstract boundary value problems Π_e are the same (or isomorphic) and all lengths ℓ_e are the same (say, $\ell_e = 1$), then we call the vector-valued quantum graph *equilateral* and $K_e = K_0$ on $\mathcal{K}_e = \mathcal{K}_0$ for all $e \in E$. A related case ($K_0 = 0$) has also been treated in [vBM13]. In the equilateral case, we have

$$\Lambda(z) = (1 \otimes (1/\sin \sqrt{z - K_0})) (1 \otimes \cos \sqrt{z - K_0} - 1 + \Delta_G \otimes 1) \quad (3.8)$$

where Δ_G denotes the (discrete) normalised Laplacian (see (2.5)) and where we have identified $\mathcal{G} \cong \ell_2(V, \deg) \otimes \mathcal{K}_0$. Since $1 \otimes (1/\sin \sqrt{z - K_0})$ is a bijective operator and $\sigma(A \otimes 1 - 1 \otimes B) = \sigma(A) - \sigma(B) = \{a - b \mid a \in \sigma(A), b \in \sigma(B)\}$, we have in particular (using (2.4) for the first equivalence)

$$\begin{aligned} \lambda \in \sigma(H) &\iff 0 \in \sigma(\Lambda(z)) \iff 0 \in \sigma(1 - \cos \sqrt{z - K_0}) - \sigma(\Delta_G) \\ &\iff \sigma(1 - \cos \sqrt{z - K_0}) \cap \sigma(\Delta_G) \neq \emptyset \\ &\iff \exists \kappa \in \sigma(K_0): 1 - \cos \sqrt{z - \kappa} \in \sigma(\Delta_G) \end{aligned}$$

provided $z \notin \sigma(H^D) = \sigma(H_{[0,1]}^D \otimes 1 + 1 \otimes K_e) = \{(n\pi)^2 + \kappa \mid n = 1, 2, \dots, \kappa \in \sigma(K_0)\}$.

We have therefore shown the following:

3.12 Corollary. Assume that all edge abstract boundary value problems Π_e are the same, i.e., associated with $(\mathfrak{k}_0, \mathcal{K}_0, [0, 1])$ (see the beginning of Section 3.4), and that all vertex spaces are standard ($\mathcal{G}_v = \{(\varphi, \dots, \varphi) \in \mathcal{K}_0^{\deg v} \mid \varphi \in \mathcal{K}_0\}$). Then the Dirichlet-to-Neumann operator is given by (3.8). Moreover, the spectrum of the vector-valued quantum graph Laplacian H is characterised by

$$\lambda \in \sigma(H) \iff \exists \kappa \in \sigma(K_0): 1 - \cos \sqrt{\lambda - \kappa} \in \sigma(\Delta_G)$$

provided $\lambda \notin \sigma(H^D) = \{(n\pi)^2 + \kappa \mid n = 1, 2, \dots; \kappa \in \sigma(K_0)\}$.

Molchanov and Vainberg [MV06] treated the asymptotic behaviour $\varepsilon \rightarrow 0$ of a Dirichlet Laplacian on the product $X \times (M, g_\varepsilon)$, where X is a metric graph, (M, g) is a compact Riemannian manifold with boundary and $g_\varepsilon = \varepsilon^2 g$. In our notation, it means that $\mathcal{H}_0 = \mathbf{L}_2(M, g_\varepsilon)$ and $K_0 = \Delta_{(M, g_\varepsilon)} = \varepsilon^{-2} \Delta_{(M, g)}$ with Dirichlet boundary conditions. It would be interesting to compare this model with the usual ε -tubular neighbourhood model with Dirichlet boundary conditions. Our methods allow such an analysis, see Section 4.

3.5 Vertex-edge coupling

Here, we treat the coupling when there are building blocks for each vertex-edge of a given graph $G = (V, E, \partial)$. Formally this coupling is just a *vertex-based coupling* for the corresponding *subdivision graph* $SG = (A, B, \tilde{\partial})$ of G (see Definition 2.1). Assume that Π_a is an abstract boundary value problem for each vertex $a \in A = V \cup E$ of the subdivision graph. The family $(\Pi_a)_{a \in A}$ allows a vertex-edge coupling if the following holds:

We say that the family of abstract boundary value problems $(\Pi_v)_{v \in V}$ allows a *vertex-edge coupling*, if the following holds:

1. We assume that $\sup_{a \in A} \|\Gamma_a\| < \infty$.
2. Assume there is a Hilbert space $\mathcal{G}_{e,v}$ for each edge $v \in V$ and $e \in E_v$ (i.e., each edge $b = (v, e)$ of the subdivision graph).

For the vertex vertices of SG assume that there is a bounded operator $\pi_{v,e}: \mathcal{G}_v \rightarrow \mathcal{G}_{e,v}$. We set $\Gamma_{v,e} := \pi_{v,e} \Gamma_v: \mathcal{H}_v^1 \rightarrow \mathcal{G}_{e,v}$. Moreover, let $\mathcal{G}_v^{\max} := \bigoplus_{e \in E_v} \mathcal{G}_e$ and $\iota_v: \mathcal{G}_v \rightarrow \mathcal{G}_v^{\max}$, $\iota_v \varphi_v = (\pi_{v,e} \varphi_v)_{e \in E_v}$. We assume that ι_v is an isometric embedding.

For the edge vertices of SG we assume that $\mathcal{G}_e = \bigoplus_{v=\partial_\pm e} \mathcal{G}_{e,v}$ (i.e., the vertex coupling is maximal at $e \in A$). We set $\Gamma_{e,v}: \mathcal{H}_e^1 \rightarrow \mathcal{G}_{e,v}$, $\Gamma_{e,v} f_e := (\Gamma_e \varphi_e)_v$.

3. For each edge $(v, e) \in B$, assume that $\text{ran } \Gamma_{v,e} = \text{ran } \Gamma_{e,v} =: \mathcal{G}_{e,v}^{1/2}$.

The formulas for the (subdivision) vertex-coupled abstract boundary value problem can be taken from Section 3.2 verbatim. Let us give two typical examples of such couplings:

Vertex-edge coupling with maximal coupling space: shrinking graph-like manifolds.

Consider a thin neighbourhood $X = X_\varepsilon$ of an embedded metric graph X_0 or a thin graph-like manifold of dimension $d \geq 2$ and decompose $X = X_\varepsilon$ into its closed vertex and edge neighbourhoods $X_v = X_{\varepsilon,v}$ and $X_e = X_{\varepsilon,e}$, respectively (see e.g. [EP05, EP09] or [P12]). We omit the shrinking parameter $\varepsilon > 0$ in the sequel, as it does not affect the coupling.

The abstract boundary value problems Π_v and Π_e are the ones for X_v and X_e with (internal) boundary ∂X_v and ∂X_e given as follows: Let $Y_{e,v} := X_e \cap X_v$ and assume that $Y_{e,v}$ is isometric with a smooth $(d-1)$ -dimensional manifold Y_e for $v = \partial_\pm e$. The (internal) boundary of X_v and X_e is now $\partial X_v = \bigcup_{e \in E_v} (X_v \cap X_e)$ and $\partial X_e = \bigcup_{v=\partial_\pm e} (X_v \cap X_e)$, respectively. We also assume that X_e is a product $I_e \times Y_e$ with $I_e = [0, \ell_e]$.

The boundary spaces for each edge $b = (v, e) \in B$ are $\mathcal{G}_{e,v} = \mathbf{L}_2(Y_{e,v}) \cong \mathbf{L}_2(Y_e)$. Note that $\mathcal{G}_v = \bigoplus_{e \in E_v} \mathcal{G}_{e,v}$, i.e., that the vertex coupling at the (original) vertices $v \in A$ is also maximal.

Under the typical uniform lower positive bound (3.5) and a suitable decomposition of X into X_v and X_e , one can show that $\|\Gamma_a\|$ is uniformly bounded. The other conditions above can also be seen easily.

If we consider again the shrinking parameter and if we assume that $X_{\varepsilon,v}$ is ε -homothetic (see example before Proposition 2.11), then (depending on the energy form and boundary conditions

chosen on X if the external boundary ∂X is non-trivial), we can apply Proposition 2.11. *Note that the boundary values of the eigenfunction on X_v corresponding to the eigenvalue 0 determine the limit boundary space, i.e., the vertex coupling appearing below as a proper subspace of \mathcal{G}_v^{\max} .*

Vertex-edge coupling with non-maximal coupling space: trivial vertex abstract boundary value problems.

Let us now construct another vertex-edge coupled abstract boundary value problem appearing e.g. in the limit situation of a shrinking graph-like space:

Assume that $(\Pi_e)_{e \in E}$ allows an edge coupling and that each abstract boundary value problem Π_e is bounded. For each $v \in V$ choose a subspace \mathcal{G}_v of \mathcal{G}_v^{\max} (e.g. given as the boundary values of the eigenfunction corresponding to 0 of a vertex neighbourhood). For the vertex abstract boundary value problems assume that $\Pi_v = (\text{id}, \mathcal{G}_v, 0, \mathcal{G}_v, \mathcal{G}_v)$, i.e., all Π_v 's are trivial (see Definition 2.2).

The corresponding vertex-edge-coupled abstract boundary value problem $\tilde{\Pi} = (\tilde{\Gamma}, \mathcal{G}, \tilde{\mathfrak{h}}, \tilde{\mathcal{H}}^1, \tilde{\mathcal{H}})$ is given as follows: The coupling condition in $\tilde{\mathcal{H}}^1$ here reads as $\Gamma_{e,v} f_e = \Gamma_{v,e} f_v$ for all $e \in V_v$ and $v \in V$. We define $\Gamma_v^{\text{int}} f := (\Gamma_{v,e} f_v)_{e \in E_v}$ and $\Gamma_v^{\text{ext}} f := (\Gamma_{e,v} f_e)_{e \in E_v}$, hence the coupling condition becomes $\Gamma_v^{\text{int}} f = \Gamma_v^{\text{ext}} f$. As $\Gamma_v^{\text{int}} f_v = (\Gamma_{v,e} f_v)_{e \in E_v} = \Gamma_v f_v = f_v \in \mathcal{G}_v$ the coupling condition is equivalent with $f_v = \Gamma_v^{\text{ext}} f \in \mathcal{G}_v$. Hence we have

$$\begin{aligned} \tilde{\mathcal{H}} &= \bigoplus_{e \in E} \mathcal{H}_e \oplus \bigoplus_{v \in V} \mathcal{G}_v, & \mathcal{G} &= \bigoplus_{v \in V} \mathcal{G}_v, \\ \tilde{\mathcal{H}}^1 &= \{ f = (f_a)_{a \in E \cup V} \in \tilde{\mathcal{H}}^{1, \text{dec}} \mid \Gamma_v^{\text{ext}} f = f_v \in \mathcal{G}_v \forall v \in V \} \\ \tilde{\mathfrak{h}} &= \tilde{\mathfrak{h}}^{\text{dec}} \upharpoonright_{\tilde{\mathcal{H}}^1}, \text{ i.e., } \tilde{\mathfrak{h}}(f) = \sum_{e \in E} \mathfrak{h}_e(f_e), & \tilde{\Gamma} f &= (\Gamma_v^{\text{ext}} f)_{v \in V} \end{aligned}$$

(the $(\cdot)^{\text{dec}}$ -labelled objects are defined as in Section 3.1). Such operators have been treated in [P12, Sec. 3.4.4] under the name “*extended operator*”.

We have the following result, showing that the vertex-edge coupling with trivial vertex abstract boundary value problems leads just to the edge coupling:

3.13 Theorem. *Let G be a discrete graph. Assume that Π is an abstract boundary value problem obtained from a family $(\Pi_e)_{e \in E}$ of abstract boundary value problems allowing an edge coupling and choose a closed subspace \mathcal{G}_v of $\mathcal{G}_v^{\max} = \bigoplus_{e \in E_v} \mathcal{G}_{e,v}$ for each vertex. As vertex family $(\Pi_v)_{v \in V}$ we choose the trivial abstract boundary value problem $\Pi_v = (\text{id}, \mathcal{G}_v, 0, \mathcal{G}_v, \mathcal{G}_v)$ for each vertex.*

Then the vertex-edge-coupled abstract boundary value problem $\tilde{\Pi}$ (i.e., vertex-coupled with respect to the subdivision graph SG) is equivalent (see below) with the edge-coupled abstract boundary value problem Π according to the original graph G .

Proof. Equivalence of two abstract boundary value problems Π and $\tilde{\Pi}$ means that there are bicontinuous isomorphisms $U^1: \mathcal{H}^1 \rightarrow \tilde{\mathcal{H}}^1$ and $T: \mathcal{G} \rightarrow \tilde{\mathcal{G}}$ such that $T\Gamma = \tilde{\Gamma}U^1$ and $\tilde{\mathfrak{h}}(U^1 f) = \mathfrak{h}(f)$. Here we have

$$\mathcal{H} = \bigoplus_{e \in E} \mathcal{H}_e, \quad \mathcal{H}^1 = \left\{ f \in \bigoplus_{e \in E} \mathcal{H}_e^1 \mid \forall v \in V: \Gamma_v^{\text{ext}} f \in \mathcal{G}_v \right\}$$

and $\tilde{\mathcal{H}}^1$ is given above. Set $U^1 f = (f, (\Gamma_v^{\text{ext}} f)_{v \in V}) =: (f, \Gamma^{\text{ext}} f)$ then $\tilde{\mathfrak{h}}(U^1 f) = \mathfrak{h}(f)$ and

$$\|U^1 f\|_{\tilde{\mathcal{H}}^1}^2 = \tilde{\mathfrak{h}}(U^1 f) + \|f\|_{\mathcal{H}}^2 + \|\Gamma^{\text{ext}} f\|_{\mathcal{G}}^2 = \|f\|_{\mathcal{H}^1}^2 + \|\Gamma^{\text{ext}} f\|_{\mathcal{G}}^2 \leq (1 + \|\Gamma^{\text{ext}}\|) \|f\|_{\mathcal{H}^1}^2$$

while for the inverse $(U^1)^{-1}(f, f_0) = f$ (with $f \in \mathcal{H}$, $f_0 \in \mathcal{G}$) we have the estimate $\|(U^1)^{-1}(f, f_0)\|_{\mathcal{H}^1}^2 = \|f\|_{\mathcal{H}^1}^2 \leq \|(f, f_0)\|_{\tilde{\mathcal{H}}^1}^2$. Moreover, we set $T\varphi = \varphi$, then $T\Gamma f = \Gamma f = \tilde{\Gamma}(U^1 f)$. \square

The previous result allows us to consider convergence of vertex-edge coupled abstract boundary value problems component-wise, even if the limit problem is only edge-coupled. A typical example is the convergence of the Laplacian on a thin ε -neighbourhood of a metric graph, to a Laplacian on the metric graph. We discuss the general convergence scheme in the next section.

4 Convergence of abstract graph-like spaces

In this section we show how one can translate convergence of building blocks into a global convergence, expressed via the coupling of abstract graph-like spaces in Section 3 and the concept of quasi-unitary equivalence resp. quasi-isomorphy for abstract boundary value problems acting in different Hilbert spaces in Section 2.3.

Fix a graph $G = (V, E, \partial)$ and let Π and $\tilde{\Pi}$ be two vertex-coupled abstract boundary value problems arising from the building blocks Π_v and $\tilde{\Pi}_v$. As we have seen in Section 3.5, the vertex coupling comprises also the vertex-edge coupled and even some edge-coupled cases.

We want to show the following: Assume that all building blocks Π_v and $\tilde{\Pi}_v$ are quasi-isomorphic then the coupled Neumann forms \mathfrak{h} and $\tilde{\mathfrak{h}}$ of the vertex-coupled abstract boundary value problems Π and $\tilde{\Pi}$ are quasi-unitarily equivalent, as well as the coupled boundary operators Γ and $\tilde{\Gamma}$ are quasi-isomorphic. We will not treat the full (natural) problem of the quasi-isomorphy of Π and $\tilde{\Pi}$ here, as we will not show that the boundary identification operators I, I' are quasi-isomomorphic (this would mean to impose additional assumptions).

One problem with the coupling is that the naively defined identification operator acts as

$$J^{1,\text{dec}} := \bigoplus_{v \in V} J_v^1 : \mathcal{H}^{1,\text{dec}} = \bigoplus_{v \in V} \mathcal{H}_v^1 \longrightarrow \tilde{\mathcal{H}}^{1,\text{dec}} = \bigoplus_{v \in V} \tilde{\mathcal{H}}_v^1$$

but it is a priori not true that $J^{1,\text{dec}}(\mathcal{H}^1) \subset \tilde{\mathcal{H}}^1$, i.e., that $J^{1,\text{dec}}$ respects the coupling condition along the different vertex building blocks as in Section 3.2. In order to correct this, we need the following definition (for the existence of such operators, see the propositions after our next theorem):

4.1 Definition. Let Π be a vertex-coupled abstract boundary value problem arising from the vertex building blocks $(\Pi_v)_{v \in V}$. We say that Π *allows a smoothing operator* if there is a bounded operator $B : \mathcal{H}^{1,\text{dec}} \longrightarrow \mathcal{H}^{1,\text{dec}}$ such that $f - Bf \in \mathcal{H}^1$ for all $f \in \mathcal{H}^{1,\text{dec}}$.

A simpler version of the following result can also be found in [P12, Sec. 4.8]:

4.2 Theorem. Let $G = (V, E, \partial)$ be a discrete graph and $(\Pi_v)_v, (\tilde{\Pi}_v)_v$ two families of abstract boundary value problems allowing a vertex coupling. Assume that Π_v and $\tilde{\Pi}_v$ are δ_v -quasi-isomorphic for each $v \in V$. Moreover, assume that $\delta := \sup_{v \in V} \delta_v < \infty$ and that the vertex-coupled abstract boundary value problems Π and $\tilde{\Pi}$ allow smoothing operators B and \tilde{B} such that

$$\|\tilde{B}J^{1,\text{dec}}f\|_{\tilde{\mathcal{H}}^{1,\text{dec}}} \leq \delta \|f\|_{\mathcal{H}^{1,\text{dec}}} \quad \text{and} \quad \|BJ^{1,\text{dec}}u\|_{\mathcal{H}^{1,\text{dec}}} \leq \delta \|u\|_{\tilde{\mathcal{H}}^{1,\text{dec}}} \quad (4.1)$$

for $f \in \mathcal{H}^1$ and $u \in \tilde{\mathcal{H}}^1$. Then \mathfrak{h} and $\tilde{\mathfrak{h}}$ are δ' -quasi-unitarily equivalent; and Γ and $\tilde{\Gamma}$ are δ' -close where $\delta' = O(\delta)$.

Proof. We define $J^1 : \mathcal{H}^1 \longrightarrow \tilde{\mathcal{H}}^1$, $J^1 := (\text{id}_{\tilde{\mathcal{H}}^1} - \tilde{B})J^{1,\text{dec}}$. From the smoothing property, we have $J^1f \in \tilde{\mathcal{H}}^1$ for any $f \in \mathcal{H}^{1,\text{dec}}$, hence J^1 maps into the right space. Similarly, we define $J'^1 := (\text{id}_{\mathcal{H}^1} - B)J'^{1,\text{dec}}$. The identification operators on \mathcal{H} and $\tilde{\mathcal{H}}$ are given as $J := \bigoplus_{v \in V} J_v$ and $J' := \bigoplus_{v \in V} J'_v$. Then we have

$$\|Jf - J^1f\|_{\tilde{\mathcal{H}}}^2 \leq 2 \sum_{v \in V} \|J_v f_v - J_v^1 f_v\|_{\tilde{\mathcal{H}}_v}^2 + 2 \|\tilde{B}J^{1,\text{dec}}f\|_{\tilde{\mathcal{H}}^{1,\text{dec}}}^2 \leq 4\delta^2 \|f\|_{\mathcal{H}^1}^2$$

using our assumptions. A similar property holds for J' and J'^1 . The other properties of Definition 2.7 for J and J' follow directly from the ones of J_v and J'_v (as in [P12, Sec. 4.8]). For the

δ -closeness of \mathfrak{h} and $\tilde{\mathfrak{h}}$ we have

$$\begin{aligned} \left| \tilde{\mathfrak{h}}(J^1 f, u) - \mathfrak{h}(f, J^1 u) \right|^2 &\leq 3 \left(\sum_{v \in V} \left| \tilde{\mathfrak{h}}_v(J_v^1 f_v, u_v) - \mathfrak{h}_v(f_v, J_v^1 u_v) \right| \right)^2 \\ &\quad + 3 \left| \tilde{\mathfrak{h}}(\tilde{B} J^{1, \text{dec}} f, u) \right|^2 + 3 \left| \mathfrak{h}(f, \tilde{B} J^{1, \text{dec}} u) \right|^2 \leq 9 \delta^2 \|f\|_{\mathcal{H}^1}^2 \|u\|_{\mathcal{H}^1}^2 \end{aligned}$$

using again (4.1). For the boundary identification operators we set $I: \mathcal{G} \rightarrow \mathcal{G}$ where $(I\varphi)_e = \frac{1}{2} \sum_{v=\partial_{\pm} e} \tilde{\pi}_{v,e} I_v \pi_{v,e}^* \varphi_e$ and similarly for I' . Then, we have the following estimates for the closeness of the boundary maps

$$\begin{aligned} \|(I\Gamma - \tilde{\Gamma} J^1) f\|_{\mathcal{G}}^2 &= \sum_{e \in E} \|((I\Gamma - \tilde{\Gamma} J^1) f)_e\|_{\mathcal{G}_e}^2 \\ &\leq \sum_{e \in E} \sum_{v=\partial_{\pm} e} \frac{1}{2} \left\| \tilde{\pi}_{v,e} (I_v \Gamma_v f_v - (\tilde{\Gamma} J^1 f)_v) \right\|_{\mathcal{G}_e}^2 \\ &\leq \frac{1}{2} \sum_{v \in V} \left\| \tilde{\pi}_v (I_v \Gamma_v f_v - (\tilde{\Gamma} J^1 f)_v) \right\|_{\mathcal{G}_v^{\max}}^2 \\ &\leq \sum_{v \in V} \left\| (I_v \Gamma_v - \tilde{\Gamma} J_v^1) f_v \right\|_{\mathcal{G}_v}^2 + \sup_v \|\tilde{\Gamma}_v\|^2 \|\tilde{B} J^{1, \text{dec}} f\|_{\mathcal{H}^1}^2 \\ &\leq \delta^2 (1 + \sup_v \|\tilde{\Gamma}_v\|^2) \|f\|_{\mathcal{H}^1}^2. \end{aligned}$$

by our assumptions. Similarly, we show the related property for $I'\tilde{\Gamma} - \Gamma J^1$.

$\delta' = \delta \max\{3, \sup_v \|\Gamma_v\| + 1, \sup_v \|\tilde{\Gamma}_v\| + 1\}$ will do the job. \square

Let us now prove the existence of smoothing operators:

4.3 Proposition. *Assume that there are operators $\chi_{e,v}: \mathcal{G}_e^{1/2} \rightarrow \mathcal{H}_v^1$ such that*

$$\Gamma_{v,e} \chi_{e,v} \varphi_e = \varphi_e, \quad \Gamma_{v,e} \chi_{e',v} \varphi_{e'} = 0, \quad e, e' \in E_v, e \neq e', v \in V.$$

Assume in addition that $C^2 = \sup_v \sum_{e \in E_v} \|\chi_{v,e}\|_{\mathcal{G}_e^{1/2} \rightarrow \mathcal{H}_v^1}^2 < \infty$ then

$$(Bf)_v := \frac{1}{2} \sum_{e \in E_v} \chi_{e,v} (\Gamma_{v,e} f_v - \Gamma_{v,e} f_{v_e})$$

defines a smoothing operator.

Proof. We have to show that $f - Bf \in \mathcal{H}^1$ whenever $f \in \mathcal{H}^{1, \text{dec}}$, but this follows immediately from the fact that $\Gamma_{v,e}(f - Bf) = \frac{1}{2} \sum_{w=\partial_{\pm} e} \Gamma_{w,e} f_w$ is independent of $v = \partial_{\pm} e$. The boundedness of $B: \mathcal{H}^{1, \text{dec}} \rightarrow \mathcal{H}^{1, \text{dec}}$ follows easily. \square

One can e.g. choose $\chi_{e,v} = S_v \pi_{v,e}^*$ under suitable assumptions on the maps $\pi_{v,e}$ (e.g., $\pi_{v,e}^*(\mathcal{G}_e^{1/2}) \subset \mathcal{G}_v^{1/2}$), where S_v is the Dirichlet solution operator of Π_v . The necessary assumptions are typically fulfilled in our graph-like manifold example; in particular, if the boundary components Y_e , $e \in E_v$, do not touch in ∂X_v , see Example 3.2.

Finally, we show the norm bound (4.1) of the smoothing operators under a slightly stronger assumption than the δ -closeness of Γ and $\tilde{\Gamma}$ (see Definition 2.6):

4.4 Proposition. Assume that a smoothing operator $\tilde{B}: \mathcal{H}^{1,\text{dec}} \rightarrow \mathcal{H}^{1,\text{dec}}$ as in Proposition 4.3 exists and that there is $\delta > 0$ such that

$$\sum_{e \in E_v} \left\| \tilde{\pi}_{v,e} (\tilde{\Gamma}_v J_v^1 - I_v \Gamma_v) f_v \right\|_{\mathcal{G}_e^{1/2}}^2 \leq \delta^2 \|f_v\|_{\mathcal{H}_v^1}^2$$

holds for all $v \in V$ and $f_v \in \mathcal{H}_v^1$. Then $\|\tilde{B} J^{1,\text{dec}} f\|_{\mathcal{H}^{1,\text{dec}}} \leq \delta C \|f\|_{\mathcal{H}^1}$ holds for all $f \in \mathcal{H}^1$.

Proof. We have

$$\begin{aligned} \|\tilde{B} J^{1,\text{dec}} f\|_{\mathcal{H}^{1,\text{dec}}}^2 &= \sum_{v \in V} \left\| \frac{1}{2} \sum_{e \in E_v} \tilde{\chi}_{e,v} (\tilde{\Gamma}_{v,e} J_v^1 f_v - \tilde{\Gamma}_{v,e} J_{v_e}^1 f_{v_e}) \right\|_{\mathcal{H}_v^1}^2 \\ &= \frac{1}{4} \sum_{v \in V} \left\| \sum_{e \in E_v} \tilde{\chi}_{e,v} \left((\tilde{\Gamma}_{v,e} J_v^1 - \tilde{\pi}_{v,e} I_v \Gamma_{v,e}) f_v + (\tilde{\pi}_{v,e} I_{v_e} \Gamma_{v_e,e} - \tilde{\Gamma}_{v,e} J_{v_e}^1) f_{v_e} \right) \right\|_{\mathcal{H}_v^1}^2 \\ &\leq C^2 \sum_{v \in V} \sum_{e \in E_v} \left\| \tilde{\pi}_{v,e} (\tilde{\Gamma}_v J_v^1 - I_v \Gamma_{v,e}) f_v \right\|_{\mathcal{G}_e^{1/2}}^2 \leq C^2 \delta^2 \|f\|_{\mathcal{H}^1}^2 \end{aligned}$$

where we used that $\Gamma_{v,e} f_v = \Gamma_{v_e,e} f_{v_e}$ for the second equality. \square

A careful observer might know that the following quote is not a rude reminder of the discomfort of aging, but just a quote from Pavel's web page . . .

Epilogue. „*Hlídejte si ty vzácné okamžiky, kdy vám to ještě myslí. Mohou být poslední. . .*“

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